High Energy Asymptotics of the Scattering Matrix for Schrödinger and Dirac Operators

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Outline of Talk

Purpose: We show that the scattering matrix, in the high energy limit, is a semiclassical pseudodifferential operator and compute its principal symbol. Then we obtain the asymptotics of the spectrum of the scattering matrix as a corollary. We consider rather general scaler equations, as well as 2 and 3 dimensional Dirac operators.

Contents:

1. Single operator

Hamilton Operators

▷ We consider a pair of $\mathbf{m}$-th order partial differential operators $\mathbf{H_0}$ and $\mathbf{H}$ on $\mathcal{H} = L^2(\mathbb{R}^d)$, defined by

$$H_0 = \sum_{|\alpha|=m} b_\alpha D_x^\alpha, \quad H = H_0 + \sum_{|\alpha|\le m-1} b_\alpha(x) D_x^\alpha,$$

where $d, m \geq 1$, $\{b_\alpha\}_{|\alpha|=m}$ are constants, $\{b_\alpha(x)\}_{|\alpha|\le m-1}$ are smooth functions, and $D_x = -i\frac{\partial}{\partial x}$.

▷ We write $p_0(\xi) = \sum_{|\alpha|=m} b_\alpha \xi^\alpha$, and we suppose the H_0 is strongly elliptic, i.e., $p_0(\xi) \geq c|\xi|^m$, $\xi \in \mathbb{R}^d$, with some $c > 0$.

▷ We suppose $\{b_\alpha(x)\}_{|\alpha|\le m-1}$ are **short-range**, i.e., there is $\mu > 1$ such that for any $\beta \in \mathbb{Z}_+^d$,

$$\left|\partial_x^\beta b_\alpha(x)\right| \leq C_\beta \langle x\rangle^{-\mu-|\beta|}, \quad x \in \mathbb{R}^d, |\alpha| < m,$$

with some $C_\beta > 0$.

▷ We also suppose H_0 and H are formally symmetric differential operators. The H_0 and H are self-adjoint with $\mathcal{D}(H) = \mathcal{D}(H_0) = H^m(\mathbb{R}^d)$.

Scattering Theory and Scattering Matrix

▷ Under these assumptions, the wave operators

$$W_\pm = \operatorname*{s\text{-}lim}_{t\to\pm\infty} e^{itH} e^{-itH_0}$$

exists and are complete: $\mathrm{Ran}[W_\pm] = \mathcal{H}_c(H)$.

▷ We recall basic properties of the wave operators:

▷ It is easy to see H_0 has absolutely continuous spectrum, and

$$HW_\pm = W_\pm H_0.$$

This intertwining property implies $\mathrm{Ran}[W_\pm] \subset \mathcal{H}_{ac}(H)$ in general.

▷ By the definition, if we set $\psi = W_\pm \psi_\pm$, $\psi, \psi_\pm \in \mathcal{H}$, then

$$e^{-itH}\psi \sim e^{-itH_0}\psi_\pm \text{ as } t \to \pm\infty$$

Thus, $W_\pm$ connects the time evolution $e^{-itH}\psi$ with the free time evolution $e^{-itH_0}\psi_\pm$ as $t \to \pm\infty$. The completeness implies that for any $\psi \in \mathcal{H}_c(H)$ has this property with some $\psi_\pm \in \mathcal{H}$.

▷ By the intertwining property, completeness also implies that $\mathcal{H}_c(H)$ is unitarily equivalent to $\mathcal{H}$, and the equivalences are given by $W_\pm$.

▷ The scattering operator is defined by $\mathbf{S} = \mathbf{W}_+^* \mathbf{W}_-$, which connects the time evolution at $\mathbf{t} = -\infty$ to that at $\mathbf{t} = +\infty$.

▷ The spectral representation of $\mathbf{H_0}$ is given by

$$\mathbf{F_0} \; : \; \mathbf{L}^2(\mathbb{R}^{\mathbf{d}}) \to \int_{\mathbb{R}_+}^{\oplus} \mathbf{L}^2(\boldsymbol{\Sigma}_{\mathbf{E}}, \mathbf{m}_{\mathbf{E}}) \mathbf{dE},$$

$$\mathbf{F_0}(\varphi)(\mathbf{E}; \cdot) = \mathcal{F}\varphi(\cdot)\big|_{\boldsymbol{\Sigma}_{\mathbf{E}}} \in \mathbf{L}^2(\boldsymbol{\Sigma}_{\mathbf{E}}, \mathbf{m}_{\mathbf{E}}), \; \mathbf{E} > \mathbf{0},$$

where $\boldsymbol{\Sigma}_{\mathbf{E}} = \left\{ \boldsymbol{\xi} \in \mathbb{R}^{\mathbf{d}} \;\middle|\; \mathbf{p_0}(\boldsymbol{\xi}) = \mathbf{E} \right\}$ and $\mathbf{dm_E} = \mathbf{dS}/|\mathbf{dp_0}(\boldsymbol{\xi})|$, $\mathcal{F}$ is the Fourier transform.

▷ By the intertwining property: $\mathbf{SH_0} = \mathbf{H_0S}$, $\mathbf{S}$ is decomposed to scattering matrices

$$\hat{\mathbf{S}}(\mathbf{E}) \; : \; \mathbf{L}^2(\boldsymbol{\Sigma}_{\mathbf{E}}, \mathbf{m}_{\mathbf{E}}) \to \mathbf{L}^2(\boldsymbol{\Sigma}_{\mathbf{E}}, \mathbf{m}_{\mathbf{E}}), \; \mathbf{E} > \mathbf{0}.$$

▷ By a change of coordinate:

$$\boldsymbol{\xi} \in \boldsymbol{\Sigma}_{\mathbf{E}} \mapsto \mathbf{E}^{-1/\mathbf{m}} \boldsymbol{\xi} \in \boldsymbol{\Sigma}_1,$$

we may consider $\hat{\mathbf{S}}(\mathbf{E})$ as an operator on $\mathbf{L}^2(\boldsymbol{\Sigma}_1, \mathbf{dm}_1)$, and we write the transformed operator by $\mathbf{S}(\mathbf{E}) \in \mathbf{B}(\mathbf{L}^2(\boldsymbol{\Sigma}_1), \mathbf{m}_1)$. (We recall $\mathbf{p_0}(\boldsymbol{\xi})$ is homogeneous.)

▷ We investigate asymptotic properties of $\mathbf{S}(\mathbf{E})$ as $\mathbf{E} \to +\infty$. In the following, we write $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}$ for simplicity.

Semiclassical Reduction

▷ We set the semiclasical parameter by

$$h = E^{-1/m},$$

and we consider the semiclassical asymptotics as $h \to +0$.

▷ We write

$$E^{-1}H = p_0(hD_x) + h\,w(h, x, hD_x), \quad E^{-1}H_0 = p_0(hD_x).$$

where

$$p_j(x, \xi) = \sum_{|\alpha|=m-j} b_\alpha(x)\xi^\alpha, \quad x, \xi \in \mathbb{R}^d, j = 0, \dots, m,$$

and

$$w(h, x, \xi) = \sum_{j=1}^{m} h^{j-1} p_j(x, \xi).$$

▷ We denote the velocity by

$$v(\xi) = dp_0(\xi) = \left(\frac{\partial p_0}{\partial \xi_1}(\xi), \dots, \frac{\partial p_0}{\partial \xi_d}(\xi)\right).$$

We note, by virtue of the Euler formula: $\xi \cdot dp_0(\xi) = mp_0(\xi)$ and the ellipticity of $p_0(\xi)$, $v(\xi) \neq 0$ for $\xi \neq 0$.

h-Pseudodifferential Calculus

▷ For $\xi \in \Sigma$, we can canonically identify $T^*_\xi\Sigma$ with the normal tangent space $v(\xi)^\perp = \{x \in \mathbb{R}^d \mid x \cdot v(\xi) = 0\}$, and we denote the cotangent bundle of Σ as

$$T^*\Sigma = \{(x,\xi) \mid \xi \in \Sigma, x \in v(\xi)^\perp \subset \mathbb{R}^d\}.$$

▷ For an h-dependent symbol $a(h,x,\xi) \in C^\infty(T^*\Sigma)$, we quantize it by

$$\begin{aligned} \mathrm{Op}_h(a)f(\xi) &= a(h,-hD_\xi,\xi)f(\xi) \\ &= (2\pi h)^{-(d-1)} \iint e^{-i(\xi-\eta)\cdot x/h} a(h,x,\eta)f(\eta)d\eta dx, \quad \xi \in \Sigma, \end{aligned}$$

with $f \in C_0^\infty(\Sigma)$ in each local coordinate.

▷ If $A = \mathrm{Op}_h(a)$, then we denote the symbol of A by $a = \mathrm{Sym}(A)$.

▷ For a weight function $K(h,x,\xi) \in C^\infty(T^*\Sigma)$, we write $a = a(h,x,\xi) \in S(K,g_0)$ (Hörmander's notation) if and only if for any $\alpha, \beta \in \mathbb{Z}_+^{d-1}$,

$$\left|\partial_x^\alpha \partial_\xi^\beta a(h,x,\xi)\right| \le C_{\alpha\beta} K(h,x,\xi)\langle x\rangle^{-|\alpha|}, \quad (x,\xi) \in T^*\Sigma, h > 0$$

with some $C_{\alpha\beta} > 0$. Here $g_0 = \langle x\rangle^{-2}dx^2 + d\xi^2$ denote a metric on $T^*\Sigma$.

2. Main Theorems

▷ We denote

$$\psi(h,x,\xi) = \int_{-\infty}^{\infty} w(h, x + tv(\xi), \xi)dt, \quad \text{for } \xi \in \Sigma, x \in v(\xi)^{\perp}.$$

We note $\psi \in S(\langle x\rangle^{-\mu+1}, g_0)$. Now we can state our main result.

Theorem 1: For sufficiently large E, $S(E)$ is an h-pseudodifferential operator on Σ (with $h = E^{-1/m}$), and the principal symbol is given by $e^{-i\psi(h,x,\xi)}$, i.e.,

$$\mathrm{Sym}(S(E)) = e^{-i\psi(h,x,\xi)} + r(h,x,\xi)$$

with $r \in S(h\langle x\rangle^{-\mu}, g)$.

▷ If we write

$$\psi_1(x,\xi) = \int_{-\infty}^{\infty} p_1(x + tv(\xi), \xi)dt, \quad \xi \in \Sigma, x \in v(\xi)^{\perp},$$

then $\psi - \psi_1 \in S(h\langle x\rangle^{-\mu+1}, g)$, and hence we have the following corollary. Note that $\psi_1(x,\xi)$ is an h-independent symbol.

Corollary 2: $\mathrm{Sym}(S(E)) = e^{-i\psi_1(x,\xi)} + r'(h,x,\xi)$ with $r' \in S(h\langle x\rangle^{-\mu+1}, g)$.

▷ We then consider the case when many terms in $\mathbf{w(h,x,\xi)}$ vanishes.

Theorem 3: Let $\mathbf{k} \in \{2, \dots, \mathbf{m}\}$ and suppose $\mathbf{p_j(x,\xi) = 0}$ for $\mathbf{j = 1, \dots, k-1}$. We write

$$\psi_k(x,\xi) = \int_{-\infty}^{\infty} p_k(x + tv(\xi), \xi)dt, \quad \xi \in \Sigma, x \in v(\xi)^{\perp},$$

and $\phi_k(h,x,\xi) = \sum_{j=k}^{m} h^{j-k}\psi_j(x,\xi)$. Then, for sufficiently large $\mathbf{E}$,

$$\mathrm{Sym}(S(E)) = e^{-ih^{k-1}\phi_k(h,x,\xi)} + r_k(h,x,\xi), \quad r_k \in S(h^k\langle x\rangle^{-\mu}, g),$$

and

$$\mathrm{Sym}(S(E)) = e^{-ih^{k-1}\psi_k(x,\xi)} + r_k'(h,x,\xi), \quad r_k' \in S(h^k\langle x\rangle^{-\mu+1}, g),$$

when $\mathbf{k < m}$.

Schrödinger operators

▷ A typical example is the Schrödinger operator. Let

$$H = \frac{1}{2}(D_x - A(x))^2 + V(x) \quad \text{on } L^2(\mathbb{R}^d), d \geq 1,$$

where $A(x) = (A_1(x), \ldots, A_d(x))$ is the vector potential and $V(x)$ is the scalar potential. We suppose $A(x)$ and $V(x)$ are smooth and satisfy our short-range assumption. In this case, Σ is the unit sphere, $v(\xi) = \xi$, and $h = E^{-1/2}$.

▷ By Corollary 2, we learn the symbol of the scattering matrix is given by

$$\exp\left(i \int_{-\infty}^{\infty} \xi \cdot A(x + t\xi)dt\right) + r_1(h, x, \xi), \quad r_1 \in S(h\langle x\rangle^{-\mu+1}, g_0).$$

▷ If $A = 0$, then we learn that the symbol of $S(E)$ is

$$\exp\left(-ih \int_{-\infty}^{\infty} V(x + t\xi)dt\right) + r_2(h, x, \xi), \quad r_2 \in S(h^2\langle x\rangle^{-\mu}, g_0)$$

by Theorem 3.

▷ Similar representation was obtained for Schrödinger operators by Yafaev (2003).

Spectral consequences

Theorem 4: Let $\psi_1 \not\equiv 0$. Then for any $\varphi \in C(\mathbb{T})$ that vanishes in a neighbourhood of the point 1, one has

$$\lim_{h\to 0} h^{d-1} \mathrm{Tr}[\varphi(S(E))] = (2\pi)^{-d+1} \int_{T^*\Sigma} \varphi(e^{i\psi_1(x,\xi)}) dx \wedge d\xi,$$

where $dx \wedge d\xi$ is the standard volume form of $T^*\Sigma$.

▷ Theorems 4 can be stated as weak convergence of the eigenvalue counting measure for $S(E)$. The proof uses functional calculus of h-pseudodifferential operators (see, e.g., [Dimassi-Sjöstrand]).

▷ This is a variation of the Weyl asymptotics. Roughly, this implies, if $J \Subset \mathbb{T}$, $J \cap \{1\} = \emptyset$, then

$$\lim_{h\to 0} h^{d-1} \#\{\text{e.v.'s of } S(E) \in J\} = (2\pi)^{-d+1} \mathrm{Vol}(\{(x,\xi) | \exp(i\psi_1(x,\xi)) \in J\}).$$

▷ If we apply this to Schrödinger operators we recover results of Bulger and Pushnitski (2013).

▷ Next, consider the case discussed in Theorem 3, i.e., $\mathbf{p_j(x,\xi) = 0}$ for $\mathbf{j = 1, \dots, k-1}$, $\mathbf{k \geq 2}$. Then we have

$$\|S(E) - I\| = O(h^{k-1}), \quad h \to 0, \quad (k \geq 2)$$

and so the spectrum of $\mathbf{S(E)}$ consists of eigenvalues located on the arc of length $\mathbf{O(h^{k-1})}$ around $\mathbf{1}$. In particular, the operator $\mathbf{Im[\log S(E)]}$ is well-defined for all sufficiently large $\mathbf{E}$.

Theorem 5: Assume the hypothesis of Theorem 3. We denote

$$B(h) = -h^{-k+1} \mathrm{Im}[\log S(E)], \quad h = E^{-1/m}.$$

Then for any $\varphi \in \mathbf{C}(\mathbb{R})$ that vanishes in a neighbourhood of $\mathbf{0}$, one has

$$\lim_{h \to 0} h^{d-1} \mathrm{Tr}[\varphi(B(h))] = (2\pi)^{-d+1} \int_{T^*\Sigma} \varphi(\psi_k(x,\xi)) dx \wedge d\xi.$$

▷ If we apply this to Schrödinger operators we recover results of Bulger and Pushnitski (2012).

3. 2 dimensional Dirac operator

Definiton of 2 dimensional Dirac operators (see, e.g., [Thaller])

▷ We set $\mathcal{H} = L^2(\mathbb{R}^2; \mathbb{C}^2) \cong L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$, and we define

$$H_0 = \sum_{j=1,2} \sigma_j D_j, \quad D_j = -i\partial_{x_j}, \ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ on } \mathcal{H}.$$

▷ Let $V(x)$, $A(x) = (A_1(x), A_2(x))$ be a real-valued functions and a vector valued function on $\mathbb{R}^2$, respectively, and we define

$$H = H_0 + W, \quad W = V(x) + \sum_{j=1,2} A_j(x)\sigma_j \quad \text{on } \mathcal{H}.$$

▷ We suppose V and A are smooth, and there is $\mu > 1$ such that for any $\alpha \in \mathbb{Z}_+^2$,

$$\left|\partial_x^\alpha V(x)\right| + \left|\partial_x^\alpha A(x)\right| \le C_\alpha \langle x\rangle^{-\mu-|\alpha|}, \quad x \in \mathbb{R}^2,$$

with some $C_\alpha > 0$. H_0 and H are self-adjoint with $\mathcal{D}(H) = \mathcal{D}(H_0) = H^1(\mathbb{R}^2; \mathbb{C}^2)$.

▷ Then it is well-known that wave operators $W_\pm(H, H_0)$ exist and they are complete: $\mathrm{Ran}[W_\pm(H, H_0)] = \mathcal{H}_{ac}(H)$. In particular, the scattering operator $S(H, H_0) = W_+(H, H_0)^* W_-(H, H_0)$ is unitary.

Construction of a spectral representation of H_0

▷ We note the symbol of $\mathbf{H_0}$ is

$$P_0(\xi) = \begin{pmatrix} 0 & \xi_1 - i\xi_2 \\ \xi_1 + i\xi_2 & 0 \end{pmatrix}, \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2.$$

This matrix is diagonalized by the matrix

$$u_0(\xi) = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta & 1 \\ -1 & \zeta^* \end{pmatrix}, \quad \zeta = \frac{\xi_1 + i\xi_2}{|\xi|},$$

and we have

$$u_0(\xi) P_0(\xi) u_0(\xi)^* = \begin{pmatrix} |\xi| & 0 \\ 0 & -|\xi| \end{pmatrix}.$$

▷ We denote by $\pi_\pm : \mathbb{C}^2 \to \mathbb{C}$ the standard projections, $\pi_+(x_1, x_2) = x_1$, $\pi_-(x_1, x_2) = x_2$, $(x_1, x_2) \in \mathbb{C}^2$. Denoting the Fourier transform by $\mathcal{F}$, we have

$$u_0(\xi) \mathcal{F} H_0 \mathcal{F}^* u_0(\xi)^* = \begin{pmatrix} |\xi| & 0 \\ 0 & -|\xi| \end{pmatrix}.$$

▷ A spectral representation of $\mathbf{H_0}$ is given by

$$F(E)\varphi(\xi) = |E|^{1/2} \pi_\pm [u_0(\xi) \mathcal{F} \varphi](|E|\xi), \quad \pm E > 0, \xi \in \mathbb{S}^1,$$

for $\varphi \in L^{2,s}(\mathbb{R}^2; \mathbb{C}^2)$, $s > 1/2$. $F(\cdot)$ is extended to a unitary equivalence:

$$F(\cdot) : \mathcal{H} \cong \int_{\mathbb{R}}^{\oplus} L^2(\mathbb{S}^1) dE.$$

▷ Using $F(\cdot)$, $S(H, H_0)$ is decomposed to the scattering matrix $S(E)$: $L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$, $E \in \mathbb{R} \setminus \{0\}$.

▷ We set $h = |E|^{-1}$ as our semiclassical parameter, and we write

$$w^{\pm}(x, \xi) = V(x) \pm \xi \cdot A(x), \; x, \xi \in \mathbb{R}^d;$$

$$\psi^{\pm}(x, \xi) = \int_{-\infty}^{\infty} w^{\pm}(x + t\xi, \xi) dt, \quad \xi \in \mathbb{S}^1, x \in T_\xi^* \mathbb{S}^1 \cong \xi^{\perp}.$$

Theorem 6: For sufficiently large $|E|$, $S(E)$ is an h-pseudodifferential operator with the symbol in $S(1, g_0)$, and the principal symbol is given by $e^{-i\psi^{\pm}(x,\xi)}$, i.e., for $\pm E \gg 0$,

$$\mathrm{Sym}(S(E)) = e^{-i\psi^{\pm}(x,\xi)} + r^{\pm}(x, \xi), \quad r^{\pm} \in S(h\langle x\rangle^{-\mu}, g_0).$$

Theorem 7 Let $\varphi \in C^\infty(\mathbb{T})$ be a function that vanishes in a neighborhood of $1 \in \mathbb{T}$. Then

$$\lim_{E \to \pm\infty} \frac{2\pi}{|E|} \mathrm{Tr}[\varphi(S(E))] = \int_{T^*\mathbb{S}^1} \varphi\big(\exp(-i\psi^{\pm}(x, \xi))\big) dx \, d\xi.$$

4. 3 dimensional Dirac operators

Definition of 3 dimensional Dirac operator

▷ We set $\mathcal{H} = L^2(\mathbb{R}^3; \mathbb{C}^4)$. The free Dirac operator is defined by

$$H_0 = \sum_{j=1}^{3} \alpha_j D_j + m\alpha_0,$$

where $m \geq 0$ and $\alpha_0, \ldots, \alpha_3$ are 4×4 Dirac matrices. The Dirac matrices satisfy the anti-commutation relations:

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} \mathbf{1}_4, \quad j, k = 0, \ldots, 3,$$

where $\mathbf{1}_4$ stands for the 4×4 identity matrix. We also write $\mathbf{1}_2$ for the 2×2 identity matrix.

▷ We choose the Dirac matrices in a standard way:

$$\alpha_0 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix}, \ \alpha_j = \begin{pmatrix} 0 & \boldsymbol{\sigma}_j \\ \boldsymbol{\sigma}_j & 0 \end{pmatrix}, \ j = 1, 2, 3,$$

where $\{\boldsymbol{\sigma}_j\}_{j=1}^3$ are the Pauli matrices:

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

▷ The perturbed Dirac operator is defined by

$$H = H_0 + W,$$

where W has the form

$$W(x) = V(x)\mathbf{1}_4 + \sum_{j=1}^{3} \alpha_j A_j(x).$$

▷ The scaler potential $V(x)$ and the vector potential $A(x) = (A_1(x), A_2(x), A_3(x))$ are real-valued smooth functions, and we suppose the short-range condition: there is $\mu > 1$ such that for any $\alpha \in \mathbb{Z}_+^3$,

$$\left|\partial_x^\alpha V(x)\right| + \sum_{j=1}^{3} \left|\partial_x^\alpha A_j(x)\right| \le C_\alpha \langle x\rangle^{-\mu-|\alpha|}, \quad x \in \mathbb{R}^3,$$

with some $C_\alpha > 0$.

▷ Under these assumptions, it is well-known that H is self-adjoint with $\mathcal{D}(H) = \mathcal{D}(H_0) = H^1(\mathbb{R}^3; \mathbb{C}^4)$, and that the wave operators

$$W_\pm(H, H_0) = \operatorname*{s-lim}_{t\to\pm\infty} e^{itH} e^{-itH_0}$$

exist and they are complete: $\mathrm{Ran}[W_\pm(H, H_0)] = \mathcal{H}_{ac}(H)$. Hence, in particular, the scattering operator $S(H, H_0) = W_+(H, H_0)^* W_-(H, H_0)$ is unitary on $\mathcal{H}$.

Diagonalization of H_0

▷ We denote the symbol of H_0 by

$$h_0(\xi) = \sum_{j=1}^{3} \xi_j \alpha_j + m\alpha_0, \quad \xi \in \mathbb{R}^3.$$

The Hermitian matrix $h_0(\xi)$ has eigenvalues

$$\pm\nu(\xi) = \pm\sqrt{|\xi|^2 + m^2},$$

and the multiplicities of $\{\pm\nu(\xi)\}$ are two for $\xi \neq 0$. We set

$$u_0(\xi) = a_+(\xi)\mathbf{1}_4 + a_-(\xi) \sum_{j=1}^{3} \hat{\xi}_j \alpha_0 \alpha_j, \quad a_\pm(\xi) = \frac{1}{\sqrt{2}}\left(1 \pm \frac{m}{\nu(\xi)}\right)^{1/2}.$$

$u_0(\xi)$ is a unitary matrix, and

$$u_0(\xi) h_0(\xi) u_0(\xi)^* = \nu(\xi)\alpha_0.$$

▷ By setting $U_0 = u_0(D_x)$, we have

$$U_0 H_0 U_0^* = \begin{pmatrix} \nu(D_x)\mathbf{1}_2 & 0 \\ 0 & -\nu(\xi)\mathbf{1}_2 \end{pmatrix},$$

and hence $\sigma(H_0) = (-\infty, -m] \cup [m, \infty)$ and the spectrum is absolutely continous.

▷ Let $\pi_\pm : \mathbb{C}^4 \to \mathbb{C}^2$ be the projections:

$$\pi_+(x) = (x_1, x_2),\ \pi_-(x) = (x_3, x_4),\ \text{for } x = (x_1, x_2, x_3, x_4) \in \mathbb{C}^4.$$

Then we have

$$\pi_\pm U_0 H_0 U_0^* \pi_\pm^* = \pm\nu(\xi)\mathbf{1}_2.$$

We note the energy surface of $\pm\nu(\xi)$ at E is given by

$$\Sigma_E = \{\xi \mid \pm\nu(\xi) = E\} = \rho\,\mathbb{S}^2, \quad \pm E > m,\ \rho = \sqrt{E^2 - m^2}.$$

▷ A spectral representation of H_0 is given by

$$F(E)\varphi(\xi) = |E|^{1/2}\rho^{1/2}\pi_\pm[\mathcal{F}U_0\varphi](\rho\xi), \quad \xi \in \mathbb{S}^2, \pm E > m,$$

for $\varphi \in H^s(\mathbb{R}^3; \mathbb{C}^4)$, $s > 1/2$, and $F(E)\varphi \in L^2(\mathbb{S}^2; \mathbb{C}^2)$.

▷ We note $F(E)H_0\varphi = EF(E)\varphi$, and $\{F(\cdot)\}$ is extended to a unitary equivalence

$$F(\cdot) : \mathcal{H} \cong \int_{\sigma(H_0)}^{\oplus} L^2(\mathbb{S}^2; \mathbb{C}^2)dE.$$

▷ Then the scattering operator $S(H, H_0)$ is decomposed to the scattering matrices $S(E) : L^2(\mathbb{S}^2; \mathbb{C}^2) \to L^2(\mathbb{S}^2; \mathbb{C}^2)$, $|E| > m$.

▷ We set $\mathbf{h} = |\mathbf{E}|^{-1}$, and we consider $\mathbf{S(E)}$ as $\mathbf{E} \to \pm\infty$. We define

$$\mathbf{W}^{\pm}(\mathbf{x}, \xi) = \mathbf{V}(\mathbf{x}) \pm \xi \cdot \mathbf{A}(\mathbf{x}), \quad \mathbf{x}, \xi \in \mathbb{R}^3,$$

and

$$\psi^{\pm}(\mathbf{x}, \xi) = \int_{-\infty}^{\infty} \mathbf{W}^{\pm}(\mathbf{x} + \mathbf{t}\xi, \xi)\mathbf{dt}, \quad \mathbf{x}, \xi \in \mathbb{R}^3, \xi \neq \mathbf{0}.$$

▷ We also define

$$\psi_1^{\pm}(\mathbf{h}, \mathbf{x}, \xi) = \psi^{\pm}(\tilde{\mathbf{x}}, \tilde{\xi}), \quad \xi \in \mathbb{S}^2, \mathbf{x} \in \mathbf{T}^*_\xi \mathbb{S}^2 \cong \xi^{\perp},$$

where

$$\tilde{\mathbf{x}} = (1 - \mathbf{h}^2\mathbf{m}^2)^{-1/2}\mathbf{x}, \quad \tilde{\xi} = (1 - \mathbf{h}^2\mathbf{m}^2)^{1/2}\xi.$$

Theorem 8: For sufficiently large $|\mathbf{E}|$, $\mathbf{S(E)}$ is a $\mathbf{2} \times \mathbf{2}$-matrix valued $\mathbf{h}$-pseudodifferential operator on $\mathbb{S}^2$ with a symbol in $\mathbf{S}(\mathbf{1}, \mathbf{g_0})$. The principal symbol is given by $\mathbf{e}^{-\mathbf{i}\psi_1^{\pm}(\mathbf{h},\mathbf{x},\xi)}\mathbf{1}_2$ for $\pm\mathbf{E} \gg \mathbf{m}$, i.e.,

$$\mathrm{Sym}[\mathbf{S(E)}] = \mathbf{e}^{-\mathbf{i}\psi_1^{\pm}(\mathbf{h},\mathbf{x},\xi)}\mathbf{1}_2 + \mathbf{r}^{\pm}(\mathbf{h}, \mathbf{x}, \xi), \quad \mathbf{r}^{\pm} \in \mathbf{S}(\mathbf{h}\langle\mathbf{x}\rangle^{-\mu}, \mathbf{g_0}).$$

▷ We note $\psi_1^{\pm}(\mathbf{h}, \mathbf{x}, \xi) \to \psi^{\pm}(\mathbf{x}, \xi)$ as $\mathbf{h} \to \mathbf{0}$, and we have

Theorem 9: Let $\varphi \in \mathbf{C}^{\infty}(\mathbb{T})$ be a function that vanishes in a neighborhood of $\mathbf{1} \in \mathbb{T}$. Then

$$\lim_{\mathbf{E}\to\pm\infty} \frac{2\pi}{|\mathbf{E}|}\mathrm{Tr}[\varphi(\mathbf{S(E)})] = 2\int_{\mathbf{T}^*\mathbb{S}^2} \varphi\left(\mathbf{e}^{-\mathbf{i}\psi^{\pm}(\mathbf{x},\xi)}\right)\mathbf{dxd}\xi.$$

5. Ideas of the proof

1. Isozaki-Kitada parametrices

We construct $\mathbf{h}$-pseudodifferential operators $\mathbf{A}_\pm = \mathbf{a}^\pm(\mathbf{h}, \mathbf{x}, \mathbf{hD_x})$ such that

$$\mathbf{H}^{\mathbf{h}}\mathbf{A}_\pm - \mathbf{A}_\pm \mathbf{H}_0^{\mathbf{h}} \sim 0$$

as $\mathbf{h} \to 0$, $|\mathbf{x}| \to \infty$ when $\pm\mathbf{x}\cdot\mathbf{v}(\xi) \geq -1+\varepsilon$, where $\mathbf{H}^{\mathbf{h}} = \mathbf{h}^{\mathbf{m}}\mathbf{H}$, $\mathbf{H}_0 = \mathbf{h}^{\mathbf{m}}\mathbf{H}_0$. We constuct $\mathbf{a}^\pm$ of the form:

$$\mathbf{a}^\pm(\mathbf{h}, \mathbf{x}, \xi) \sim e^{i\psi_\pm(\mathbf{h}, \mathbf{x}, \xi)}(1 + \mathbf{a}_1^\pm(\mathbf{h}, \mathbf{x}, \xi) + \mathbf{a}_2^\pm(\mathbf{h}, \mathbf{x}, \xi) + \cdots),$$

where

$$\psi_\pm(\mathbf{h}, \mathbf{x}, \xi) = \int_0^{\pm\infty} \mathbf{w}(\mathbf{h}, \mathbf{x} + \mathbf{t}\mathbf{v}(\xi), \xi)\mathbf{dt},$$

and $\mathbf{a}_j^\pm \in \mathbf{S}(\mathbf{h}^j\langle \mathbf{x}\rangle^{-\mu+1-j})$, $j = 1, 2, \ldots$.

2. Microlocal resolvent estimate (in the semiclassical form)

We characterize the frequency sets of the distribution kernels of $(\mathbf{H}^{\mathbf{h}} - \lambda \mp i0)^{-1}$ (in the Fourier space).

3. Representation formula of the scattering matrix

We use a variation of representation formulas of S-matrix due to Isozaki-Kitada (1986), Yafaev (2003).

4. Microlocal diagonalization of $\mathbf{H}^{\mathbf{h}}$

For Dirac operators, we use an approximate (block) diagonalization of the matrix-valued $\mathbf{h}$-pseudodifferential operator $\mathbf{H}^{\mathbf{h}}$, which is a generalization of Taylor (1975), and Helffer-Sjöstrand (1990).

Remark: We may generalize our results to more general systems, but we need to assume technical assumptions to carry out this block diagonalization argument, and probably it would be technically complicated.