Stark-Wannier ladders and cubic exponential sums

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Outline

The Stark-Wannier Hamiltonian

On $L^2(\mathbb{R})$, consider

$$H_\varepsilon = -\frac{d^2}{dx^2} + v(x) - \varepsilon x \tag{1}$$

where

- v is a real analytic 1-periodic function,
- ε is a positive constant.

Model for

- One dimensional Bloch electron in a constant electric field,
- Electrons in semi-conductor superlattices,
- Cold atoms in optical lattices.

Spectral theory:

- The spectrum of H_ε is $\mathbb{R}$ and purely absolutely continuous.
- Resolvent $(z - H_\varepsilon)^{-1}$ can be continued from upper half plane $\mathbb{C}^+$ to $\mathbb{C}^-$.
- Resonances: poles of resolvent.
- Stark-Wannier ladders : resonances form ε-periodic lattices parallel to $\mathbb{R}$ as $\tau_{-1}(z - H_\varepsilon)^{-1}\tau_1 = (z + \varepsilon - H_\varepsilon)^{-1}$ for $\mathrm{Im}\, z > 0$.

Known results

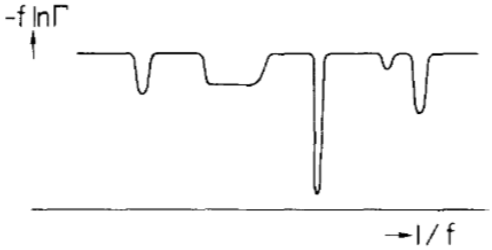
Many works (both physics and maths).
Proofs of existence of resonances for various regimes of v and ε.
Determination of the ladders when ε small: semi-classics (physics, Buslaev et al.).
Analysis only done for v finite gap potential.

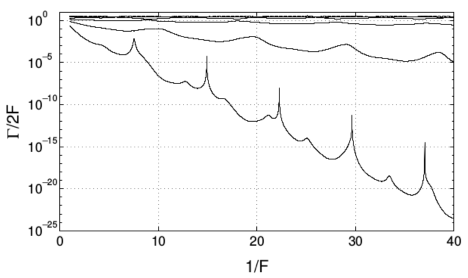
Effective Hamiltonian: $E_j(\kappa) - \xi = E$, E_j band functions for $-\frac{d^2}{dx^2} + v$,

Spectral band $\to$ one ladder: $\mathrm{Re}\, z_{j,k} = k\varepsilon - \frac{1}{2}\int_0^{2\pi} E_j(\kappa)d\kappa + O(\varepsilon)$, $k \in \mathbb{Z}$,

Imag. part exp. small: when n gaps $-\log|\mathrm{Im}\, z_{j,k}| = \frac{1}{\varepsilon}\sum_{j\le l\le n}\int_{\gamma_l} E_l(\kappa)d\kappa + O(1)$;

Resonance phenomenon between ladders: very complicated (multifractal) behavior of resonances in ε small. Dependent on number theor. prop. of ε (Avron).





Interactions of ladders. [Avron 1982 / Glück et al. 2002]

The reflection coefficient

For v entire, consider

$$-\psi''(x)+(v(x)-\varepsilon x)\psi(x)=E\psi(x),\quad x\in\mathbb{C}; \tag{2}$$

for $E\in\mathbb{C}$, $\psi_\pm$ unique solutions to (2) s.t.

$$\psi_-(x,E)=\frac{1}{\sqrt[4]{-\varepsilon x-E}}e^{-\int_x^{-E/\varepsilon}\sqrt{-\varepsilon t-E}\,dt+o(1)},\quad x\to-\infty,$$
$$\psi_+(x,E)=\frac{1}{\sqrt[4]{\varepsilon x+E}}e^{i\int_{-E/\varepsilon}^{x}\sqrt{\varepsilon t+E}\,dt+o(1)},\quad x\to+\infty.$$

Define $\psi_+^*(x,E)=\overline{\psi_+(\overline{x},\overline{E})}$.
The solutions ψ_+ and ψ_+^* linearly independent; thus

$$\psi_-(x,E)=w(E)\psi_+^*(x,E)+w^*(E)\psi_+(x,E),\quad x\in\mathbb{R}, \tag{3}$$

$w(E)$ is independent of x and the function $E\mapsto w(E)$ is entire and ε-periodic.
The reflection coefficient: $r(E)=w^*(E)/w(E)$.
The reflection coefficient is an ε-periodic meromorphic function of E.
The reflection coefficient is analytic in $\mathbb{C}_+$, and, for $E\in\mathbb{R}$, one has $|r(E)|=1$.
Poles of r = resonances of H_ε.

The Fourier coefficients of the inverse of r when $v(x)=\cos(2\pi x)$

We now assume $v(x)=\cos(2\pi x)$ and, for $\operatorname{Im}E\le 0$, define $\displaystyle\frac{1}{r(E)}=\sum_{m\in\mathbb{Z}}e^{2\pi miE/\varepsilon}p(m)$.

Theorem

Let $v(x)=2\cos(2\pi x)$. Then, as $m\to\infty$,

$$p(m)=a(\varepsilon)\sqrt{m}\,e^{-2\pi i\omega m^3-2m\log(2\pi m/e)+\delta(m)},\quad a(\varepsilon)=\sqrt{\frac{2}{\varepsilon}}\,\pi e^{i\pi/4},\quad \omega=\left\{\frac{\pi^2}{3\varepsilon}\right\},$$

where, for x real, $\{x\}$ denotes the fractional part of x, and $\delta(m)=O(\log^2 m/m)$. This estimate is locally uniform in $\varepsilon>0$.

When $\operatorname{Im}E\to-\infty$, $\displaystyle\frac{1}{r(E)}\approx a(\varepsilon)\,\mathscr{P}(E/\varepsilon)$ where

$$\mathscr{P}(s):=\sum_{m\ge1}\sqrt{m}\,e^{-2\pi i\omega m^3-2m\log(2\pi m/e)+2\pi ims}.$$

$\mathscr{P}$: mollified version of the cubic exponential sums $\displaystyle\sum_{n=1}^{N}e^{-2\pi i\omega n^3}$.

For rational frequencies ω, we get

Theorem

Let $v(x) = 2\cos(2\pi x)$. Assume that $\omega = \frac{p}{q} \in \mathbb{Q}$ with $0 \le p < q$ co-prime. For $\xi \in \mathbb{R}$, we set $I_q(\xi) := \{m \in \mathbb{Z} : |\xi - \frac{m}{q}| \le 1/2\}$. As $\operatorname{Im} E \to -\infty$, one has

$$r^{-1}(E) = \frac{b(\varepsilon)\rho}{q} \sum_{m \in I_q(\xi)} S_q(p,m) e^{\rho e^{i\pi(\xi - m/q)} + i\pi(\xi - m/q) + O(\log^2 \rho/\rho)} + e^{O(\frac{\rho}{\ln \rho})}, \qquad (4)$$

where $b(\varepsilon) = \pi^{\frac{3}{2}} e^{i\pi/4}/\sqrt{2\varepsilon}$, $\xi = \operatorname{Re} E/\varepsilon$, $\rho = e^{-\pi \operatorname{Im} E/\varepsilon}$, and

$$S_q(p,m) = \sum_{l=0}^{q-1} e^{-2\pi i \frac{pl^3 - ml}{q}}.$$

The error estimates are locally uniform in $\varepsilon > 0$.

$S_q(p,m)$ complete exponential sums.

Some facts on complete exponential sums. We assume $\omega \in \mathbb{Q}^*$.
One easily checks

Lemma

For any $q \in \mathbb{N}$ and $p \in \mathbb{N}$, $\displaystyle\sum_{m=0}^{q-1} |S_q(p,m)|^2 = q^2.$

For large q, many non-zero values $S_q(p,m)$: more precisely

Lemma

There exists a constant $C > 0$ such that, for any co-prime $q > p > 0$, one has $\#\{0 \le m < q : S_q(p,m) \ne 0\} \ge Cq^{\frac{2}{3}}$.

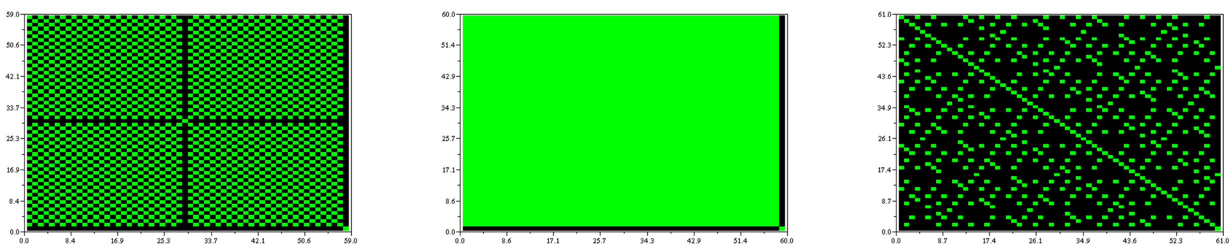


Figure : Complete cubic sums for $q = 58, 59, 60$.

The low lying resonances when $\omega \in \mathbb{Q}^*$

If $S_q(p,m)$ non zero for at least two values of m

Corollary

Assume $S_q(p,m_1) \neq 0$ and $S_q(p,m_2) \neq 0$ for $0 \leq m_1 < m_2 < q$ and $S_q(p,m) = 0$ for all $m_1 < m < m_2$. Then, for sufficiently large $y > 0$, there are resonances in the vertical half-strip

$$\left\{E \in \mathbb{C} : -\frac{\operatorname{Im} E}{\varepsilon} \geq y,\ \frac{m_1}{q} \leq \frac{\operatorname{Re} E}{\varepsilon} \leq \frac{m_2}{q}\right\},$$

and these resonances are given by: for $k \in \mathbb{N}$ large

$$\frac{E}{\varepsilon} = -i\left(\frac{\ln(\pi k)}{\pi} - \ln\sin\frac{\pi(m_2 - m_1)}{q}\right) + \frac{m_2 + m_1}{q} + o(1) \tag{5}$$

where $o(1) \underset{k\to+\infty}{\longrightarrow} 0$.

Representation (5) not uniform in q, p, m_1 and m_2:

uniformity requires $\frac{q \ln k}{k}$ and $\frac{1}{k}\ln\left|\frac{S_q(p,m_1)}{S_q(p,m_2)}\right|$ remain small.

The low lying resonances when $\omega = 0$

Assume first $\omega = 0$.

By Theorem 2.2, $(b(\varepsilon)r(E))^{-1} = \sqrt{z}\,e^{\sqrt{z}+O(\frac{\ln^2 z}{\sqrt{z}})} + e^{O(\frac{\sqrt{z}}{\ln z})}$, $z = e^{2i\pi E/\varepsilon}$.

Let $B_\varepsilon = \{E \in \mathbb{C} :\ \operatorname{Im} E \leq 0, 0 \leq \operatorname{Re} E \leq \varepsilon\}$ (as resonances on ε periodic ladders).

Above representation implies

Corollary

Assume $\omega = 0$. the resonances located in B_ε have the following properties :

- *there exists $C > 0$ s.t., for sufficiently large $y > 0$, the resonances in $B_\varepsilon \cap \{\operatorname{Im} E < -\varepsilon y\}$ are located in the domain $|\operatorname{Re} E - \varepsilon/2| \leq C\varepsilon^2/|\operatorname{Im} E|$;*
- *let $n(y)$ be the number of resonances in the rectangle $[0,\varepsilon] - i[0,\varepsilon y]$; one has*

$$n(y) = \frac{1}{\pi} e^{\pi y + o(1)} \quad \text{as} \quad y \to \infty.$$

When $\omega = 0$, difficult to describe resonances: it is not an asymptotic problem. Indeed, in a neighborhood of the line $\left\{\frac{\operatorname{Re} E}{\varepsilon} = \frac{1}{2} \operatorname{mod} 1\right\}$, resonances determined by first Fourier coefficients of $1/r$.

The main steps of the proof

Recall eq. (2) $-\psi''(x) + (v(x) - \varepsilon x)\psi(x) = E\psi(x), \quad x \in \mathbb{C}$; and want to construct Buslaev's solutions with asymptotics

$$\psi_-(x,E) = \frac{1}{\sqrt[4]{-\varepsilon x - E}} e^{-\int_x^{-E/\varepsilon} \sqrt{-\varepsilon t - E}\,dt + o(1)}, \quad x \to -\infty,$$
$$\psi_+(x,E) = \frac{1}{\sqrt[4]{\varepsilon x + E}} e^{i\int_{-E/\varepsilon}^x \sqrt{\varepsilon t + E}\,dt + o(1)}, \quad x \to +\infty.$$

$\psi(x,E)$ solution $\Longrightarrow \psi(x+1, E-\varepsilon)$ also solution.

Consistent solutions: $\psi(x+1, E-\varepsilon) = \psi(x,E)$.

Asymptotics of Buslaev solutions are consistent.

So find consistent solutions with Buslaev asymptotics.

Monodromy matrix: pick $\psi(x,E)$ solution so that $\psi(x+1,E-\varepsilon)$ linearly independent; define M

$$\begin{pmatrix} \psi(x+1,E-\varepsilon) \\ \psi(x+2,E-2\varepsilon) \end{pmatrix} = M(E) \begin{pmatrix} \psi(x,E) \\ \psi(x+1,E-\varepsilon) \end{pmatrix} \quad \Longrightarrow \quad M(E) \sim \begin{pmatrix} 0 & 1 \\ -1 & d(E) \end{pmatrix}.$$

Monodromy equation: solve $V(E-\varepsilon) = M(E)V(E)$, V 2×2 matrix $\Longrightarrow$
$\begin{pmatrix} \Psi(x,E) \\ \Psi(x+1,E-\varepsilon) \end{pmatrix} := V^{-1}(E) \begin{pmatrix} \psi(x,E) \\ \psi(x+1,E-\varepsilon) \end{pmatrix}$ consistent basis.

WKB analysis: used to construct the monodromy basis i.e. to compute d. This uses special form of periodic potential.
Should work for trigonometric polynomials as potential.

Solve monodromy equation: $M(E) \sim \begin{pmatrix} 0 & 1 \\ -1 & d(E) \end{pmatrix}$
Transfer matrix of a discrete Schrödinger equation $\Longrightarrow$ solved using continued fraction or discrete Ricatti equation depending on asymptotic regime.

Yields asymptotics of Buslaev's solution at large negative energies$\Longrightarrow$ asymptotics of the reflection coefficient.

Analysis of the reflection coefficient when $\omega \in \mathbb{Q}$:
renormalization: up to error terms, apply Poisson formula as in number theory for exponential sums. Difficulty: efficient analysis of the error terms.

Analysis of the resonances when $\omega \in \mathbb{Q}$:

- when $\omega = p/q \neq 0$: keep two principal terms in series and solve;
- when $\omega = 0$: estimate growth and use Levin-Cartan estimates for the zeros of analytic functions.