Dispersion for the wave and the Schrödinger equations outside a strictly convex obstacle and counterexamples

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Fermat's Principle :

How does light travel?
Fermat (1662): the path of a ray of light between two points must be such that the time occupied in the passage is a minimum. (Fermat's Principle)

Pour. . . trouver la véritable raison de la réfraction. . . si nous voulions employer dans cette recherche ce principe si commun et si établi, que la nature agit toujours par les voies les plus courtes, nous pourrions y trouver facilement notre compte.

In modern language, the principle says light rays follow geodesics when light travels through a medium with variable refractive index.

Light is an electromagnetic wave (Maxwell 1862)

Mathematical description of waves, at least to first approximation, is the same in many different settings.

- The scalar wave equation is :

$$\Box u = (\partial_{tt}^2 - \Delta_{\mathbb{R}^d})u = 0, \qquad \Delta_{\mathbb{R}^d} = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}.$$

  In addition to light, the same equation (approximatively) describes sound waves, small amplitude water waves, etc.

- A close relative is the Schrödinger equation (especially in the "semi-classical" setting $\hbar \to 0$):

$$\frac{\hbar}{i}\partial_t v - \frac{\hbar^2}{2}\Delta v = 0, \quad \hbar = \frac{h}{2\pi}.$$

Wavefront set

What about Fermat's last principle for a wave ?

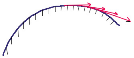
An elegant answer comes from studying the regularity of solutions to the wave equations.

Definition: Let f be a function on $\mathbb{R}^d$. A point x_0 is in the singular support of f if there is no neighbourhood of x_0 on which $f \in C^\infty$. The wavefront set measures *where* f is singular and *in what direction*:

$$WF(f) \subset T^*\mathbb{R}^d, \quad \pi(WF(f)) = \text{sing-supp f}.$$

Say we impose Dirichlet boundary conditions to the wave equation. What happens with the WF set that reaches the boundary of an obstacle?

- Transverse reflection : "angle of incidence equals angle of reflection" (billiard ball).
- Tangent to a convex obstacle : can a ray carrying WF tangent to a convex obstacle stick to it and re-release it in the "shadow region" or it simply pass on by?



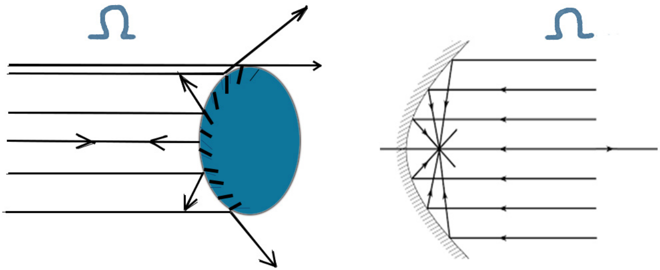
Theorem

(Melrose-Taylor 1975) No propagation into the "shadow region". If we measure analytic singularities, this becomes false.

Theorem

(Hargé-Lebeau 1994 - Keller's conjecture for C^∞ boundary) The decreasing rate in the shadow region is of the form $e^{-C\tau^{1/3}}$.

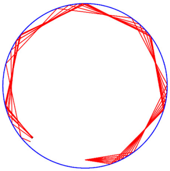
Optical rays

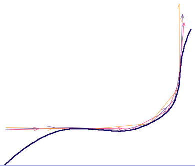


Inside a parabola : the wave shrinks in size at the focus and its L^∞ norm increases.

Glancing

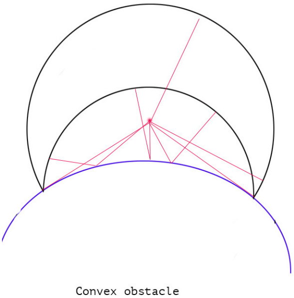
- Propagation inside a strictly convex: (any (small) piece of geodesic tangent to $\partial\Omega$ is exactly tangent at order 2 and lies outside Ω). In this case, highly - multiply reflected rays and their limits (I., Lebeau, Planchon - Annals 2014, general convex case 2015)
- Tangency, no convexity : in this case, if the ray has infinite order tangency with the boundary, even deciding what should constitute the continuation of a ray striking the boundary is difficult... (example of Taylor of many possible continuations of a given ray that hits $\partial\Omega$ with infinite order tangency).



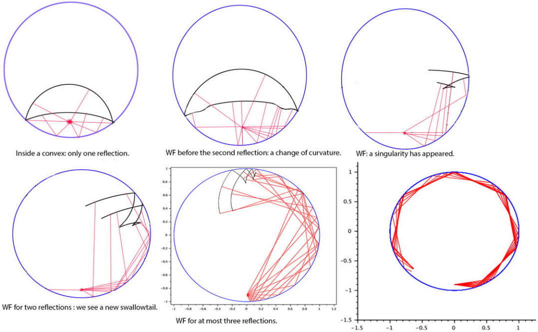


WF in the exterior/ interior of a strictly convex

Wave front outside a strictly convex obstacle



Wave front inside a convex obstacle (for at most three reflections)



A caustic/cluster point/ singularity (where light is singularly intense) in the WF should yield loss in dispersion . What *is* dispersion?

- The wave flow :

$$\sup \left| \chi(hD_t) e^{\pm it\sqrt{|\triangle_{\mathbb{R}^d}|}}(\delta_{Q_0}) \right| \leq Ch^{-d} \min(1, (\frac{h}{t})^{\frac{d-1}{2}})$$

- The classical Schrödinger flow :

$$\sup \left| \chi(hD_t) e^{\pm it\triangle_{\mathbb{R}^d}}(\delta_{Q_0}) \right| \leq Ct^{-d/2}, \quad t \geq h$$

and semi-classical

$$\sup \left| \chi(hD_t) e^{\pm i\frac{t}{h}\triangle_{\mathbb{R}^d}}(\delta_{Q_0}) \right| \leq \frac{C}{h^d} \min(1, (\frac{h}{t})^{\frac{d}{2}}).$$

"Heuristical proof" of dispersion in $\mathbb{R}^d$

- The data: Dirac $\delta_{x=x_0}$ spectrally localised at frequency $\frac{1}{h}$:

$$u_0(x) = h^{-d} \int_{\mathbb{R}^d} e^{\frac{i}{h}\xi(x-x_0)}\psi(|\xi|)d\xi$$

is concentrated in the ball $B(x_0, h)$;

- The corresponding wave flow $e^{it\sqrt{|\Delta_{\mathbb{R}^d}|}}(\delta_{x=x_0})$ at time $t > 0$,

$$u(x,t) = h^{-d} \int_{\mathbb{R}^d} e^{i\frac{t}{h}|\xi|}\psi(|\xi|)e^{\frac{i}{h}\xi(x-x_0)}d\xi$$

essentially "lives" in $S(t, x_0, h) = \{y \in \mathbb{R}^d | \quad \mathrm{dist}(y, |x - x_0| = t) \leq h\}$

- using energy conservation (for $t = 1$)

$$\|h^{-d}\mathbb{I}_{B(x_0,h)}\|_{L^2} \simeq \|u_0\|_{L^2} \simeq \|u(t=1)\|_{L^2} \simeq \|C\mathbb{I}_{S(t,x_0,h)}\|_{L^2}$$

we obtain $C = h^{-\frac{d+1}{2}}$.

Theorem

(I. - Lebeau, 2015) Θ_d = *strictly convex obstacle in* $\mathbb{R}^d$, $\Omega_d = \mathbb{R}^d \setminus \Theta_d$.

- *If* $d = 3$, *the dispersion estimates (for the wave and the Schrödinger equations inside* Ω_3*) hold true.*
- *If* $d \geq 4$, $\Theta_d = B_d(0,1) \subset \mathbb{R}^d$, *at the Poisson spot they fail.*

Recall : Strichartz do hold like in $\mathbb{R}^d$ for both wave (Smith-Sogge 1995) and (classical) Schrödinger (I. 2010) equations, $\forall d \geq 2$.

Having the full dispersion for Schrödinger in $3D$:

- we can get the endpoint Strichartz estimate.
- might also help in dealing with non-linear equation (see recent work by Killip, Visan, Zhang 2014 on the energy-critical NLS in $3D$).

Estimates at the Poisson spot

$Q_{\pm}(s)$ = source/observation points at (same) distance s from the ball, symmetric w.r.t. the center of the ball :

- Wave flow : $s = \gamma h^{-1/3}$, $t \simeq h^{-1/3}$

$$|(\chi(hD_t)e^{ih^{-1/3}\sqrt{|\Delta|}}(\delta_{Q_-})|(Q_+) \simeq \frac{1}{h^d}\left(\frac{h}{h^{-1/3}}\right)^{-\frac{d-1}{2}} h^{-\frac{d-3}{3}},$$

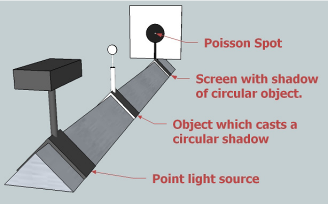
- Classical Schrödinger flow: $s \simeq \gamma h^{-1/6}$, $t \simeq h^{1/3}$

$$|(\chi(hD_t)e^{ih^{1/3}\Delta}(\delta_{Q_-})|(Q_+) \simeq (h^{1/3})^{-\frac{d}{2}} h^{-\frac{d-3}{6}},$$

- Semi-classical Schrödinger : $t \simeq h^{-2/3}$

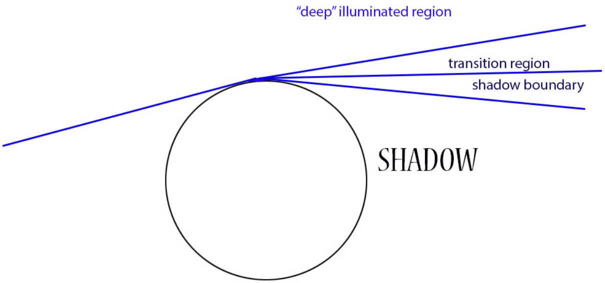
For $d \geq 4$ this contradicts the usual (flat) estimates !

Poisson - Arago spot



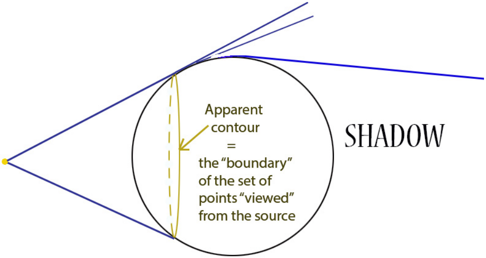
When light shines on a circular obstacle, Huygens's principle says that every point of the obstacle acts as a new point source of light. The light coming from points on the circumference of the obstacle and going to the centre of the shadow travel exactly the same distance and give rise to a bright spot at the shadow's center.

Diffraction $\Omega = \mathbb{R}^d \setminus \Theta$, Θ strictly convex



- Coordinates : $(x, y) \in \mathbb{R}_+ \times \partial\Omega$ such that $x \to (x, y)$ is the ray orthogonal to $\partial\Omega$ at y. Then $\forall Q \in \Omega$, $Q = y + xn_y$, where n_y is the outward unit normal to $\partial\Omega$ pointing towards Ω.

Glancing : $h^{1/3}$ neighbourhood of the apparent contour



- "Glancing" : a $h^{1/3}$ neighbourhood of the apparent contour.

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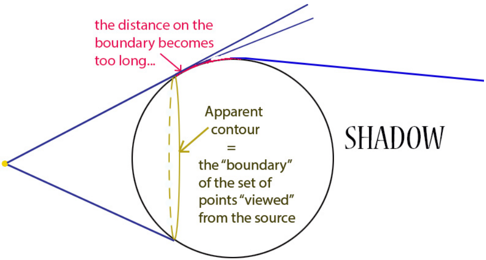
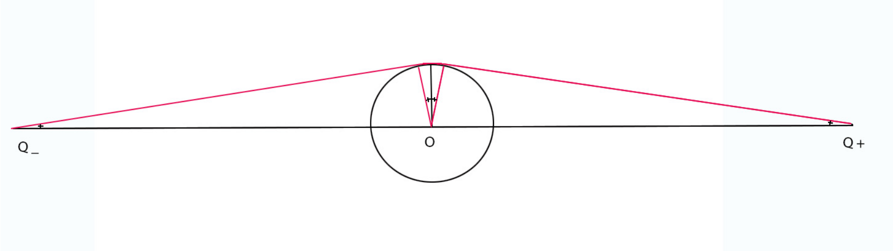


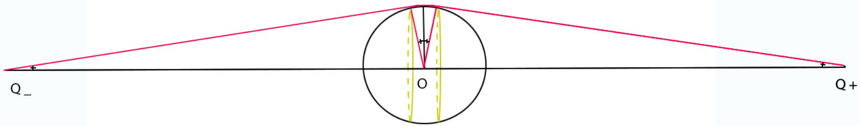
- "Glancing" : a $h^{1/3}$ neighbourhood of the apparent contour.
- If the source is not sufficiently far from Θ, in order to re-focalise, the wave would need to spend a longer time on the boundary ($\gg h^{1/3}$), but then we use Hargé-Lebeau.

Glancing : $h^{1/3}$- neighbourhood of the apparent contour



- To obtain re-focalisation we need to put the source at distance $h^{-1/3}$.

Glancing : $h^{1/3}$- neighbourhood of the apparent contour



- The distance between the apparent contour of Q_- and the apparent contour of Q_+ is $\simeq h^{1/3}$ (*allowed* distance, not exponential decay like for the shadow region !).

How do we construct the parametrix ?

Let $\Theta = B_d(0,1) \subset \mathbb{R}^d$:

- if polar coordinates : problems when the source point $\to \infty$;
- anyway not useful in the general case, so forget it ....
- Melrose-Taylor parametrix : OK, but it is defined only near $\partial\Omega$
- To handle this, use the fundamental solution and the Neumann operator to reduce to estimations for (in $d = 3$):

$$\int \int_{P \in \partial\Omega} (\partial_x u_{free}|_{\partial\Omega} - N(u_{free}|_{\partial\Omega}))(s, P) \frac{\delta(t - s - |Q - P|)}{4\pi|P - Q|} d\sigma(P) ds$$

- use NOW Melrose-Taylor (in the glancing region) : it gives the form of the left factor in terms of Airy functions.

Form of the parametrix

- For positive results ($d = 3$) forget the time : $t \simeq |Q_- - P| + |P - Q_+|$ is uniformly bounded w.r.t. $d(Q_\pm, \partial\Omega)$ (Melrose 1979 : the outgoing solution decays exponentially if time is larger than the escape time);
- Melrose-Taylor: $\exists \theta, \zeta$ phase functions which solve the eikonal equation near glancing, $\theta_0 = \theta|_{\partial\Omega}$ defines a canonical relation ($\theta(y, \eta) = y\eta$ outside the ball), $\zeta_0 = \zeta|_{\partial\Omega}$ doesn't depend on the tangential variable s.t. the left factor is

$$\tau^{d-1+\frac{2}{3}} \int e^{i\tau\theta_0(P,\eta)} \frac{\hat{F}_\tau(\tau\eta, Q_0)}{A_+(\tau^{2/3}\zeta_0(\eta))} d\eta,$$

$$\hat{F}_\tau(\tau\eta, Q_0) = \left(\frac{\tau}{|Q_- P_0|}\right)^{\frac{d-1}{2}} \tau^{-1/3} e^{i\tau(\theta_0(P_0,\eta) - |Q_0 - P_0|)},$$

where $P_0 \in$ the apparent contour $\mathcal{C}_{Q_0}$ of Q_0.

Details... for the exterior of the ball

- Melrose-Taylor in the glancing region :

$$u_{free}|_{\partial\Omega}(\tau, P, Q_0) = \left(\frac{\tau}{|P - Q_0|}\right)^{\frac{d-1}{2}} e^{-i\tau|P-Q_0|}\gamma(\tau, P, Q_0)$$
$$= Op(e)(\mathcal{F}^{-1}(A(\tau^{2/3}\zeta_0)\mathcal{F}(F(., Q_0)),$$

  e elliptic symbol.
- To get F : degenerate critical points of order 2 on $\mathcal{C}_{Q_0}$.
- In the formula to estimate : a new, similar phase, hence a second degenerate stationary phase with critical points on $\mathcal{C}_Q$, at distance at most $\gamma h^{1/3}$ from $\mathcal{C}_{Q_0}$.
- If $Q_0 = Q_-(\gamma h^{-1/3})$, $Q = Q_+(\gamma h^{-1/3})$ and $t \simeq h^{-1/3}$ then the announced estimates.

Questions

- What about the generic case?
- A "twisted sphere" still yields a counterexample...
- classify the obstacles for which dispersion holds in every d?