Distorted plane waves in chaotic scattering

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Euclidean near infinity manifold

(X, g) Riemannian manifold such that

$$(X \backslash X_0, g) \cong ((\mathbb{R}^d \backslash B(0, R)), g_{flat}),$$

with X_0 compact, $R > 0$.

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with X_0 compact, $R > 0$.
For $\xi \in \mathbb{S}^d$, define

$$E_h^0(x; \xi) := e^{i \frac{x \cdot \xi}{h}} \text{ if } x \in X \backslash X_0, \quad 0 \text{ otherwise.}$$

Definition of distorted plane waves

$$E_h(\cdot;\xi) = (1-\chi_0)E_h^0(\cdot;\xi) + E_h^1(\cdot;\xi),$$

where χ_0 is a smooth function equal to one in the interaction region X_0, and

$$E_h^1(\cdot;\xi) := -R_h[h^2\Delta_g, \chi_0]E_h^0(\cdot;\xi).$$

$$R_h := (h^2\Delta_g - (1+i0)^2)^{-1}.$$

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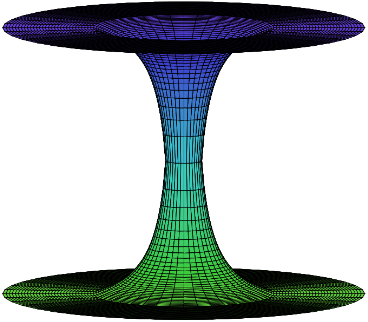
We have

$$(h^2\Delta_g - 1)E_h = 0.$$

E_h is called a *distorted plane wave* or an *Eisenstein function*.

More general structure at infinity



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Classical dynamics

Geodesic flow:

$$\Phi^t : T^*X \longrightarrow T^*X.$$

Incoming/ Outgoing tails:

$$\Gamma^{\pm} := \{\rho \in S^*X; \{\Phi^t(\rho), \pm t \leq 0\} \text{ is a bounded subset of } S^*X\}.$$

Trapped set:

$$K := \Gamma^+ \cap \Gamma^-.$$

Warning : (E_h) might not be bounded in L^2_{loc} uniformly with h!
Recall that

$$E_h(\cdot;\xi) = \chi_0 E_h^0(\cdot;\xi) - R_h[h^2\Delta_g, \chi_0]E_h^0(\cdot;\xi).$$

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If $K = \emptyset$, then (Burq 2000, Vodev 2002)

$$\|\chi R_h \chi\|_{L^2\to L^2} \leq \frac{C}{h}.$$

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If $K \neq \emptyset$, then (Bony-Burq-Ramond 2012)

$$\|\chi R_h \chi\|_{L^2\to L^2} \geq \frac{C|\log h|}{h}.$$

Micro-local limits in a general setting

Theorem (Dyatlov-Guillarmou 2013)

Suppose K has zero Liouville measure. Then for almost every $\xi \in \partial\overline{X}$, there exists a Radon measure μ_ξ on S^*X such that for each $a \in C_c^\infty(T^*X)$, we have

$$\lim_{h\to 0} h^{-1}\Big\| \langle Op_h(a)E_h(\lambda\xi), E_h(\lambda\xi)\rangle_{L^2(X)} - \int_{S^*X} a \mathrm{d}\mu_\xi \Big\|_{L^1_{\xi,\lambda}(\partial\overline{X}\times[1,1+h])} = 0.$$

We have $\mathrm{d}\mu_\xi = \lim_{t\to\infty} (\Phi^t)_*(|(1-\chi_0(x))|^2 dx d_{\{p=\xi\}})$.

Hyperbolic plane waves

Consider $X = \mathbb{H}^d$ in the half-space model.
Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^d$. Then for all $\xi \in \partial\mathbb{H}^d$,

$$E_h^0(z;\xi) := \Big(\frac{z_1}{|z-\xi|^2}\Big)^{1/2-i/h}$$

is the incoming wave from direction ξ.
It satisfies

$$\Big(-h^2\Delta - \frac{(d-1)^2}{4}\Big)E_h^0 = E_h^0.$$

The case of convex co-compact hyperbolic manifolds

$X = \Gamma \backslash \mathbb{H}^d$, Γ convex co-compact, $\xi \in \partial X$.

$$E_h(x;\xi) = \sum_{\gamma \in \Gamma} E_h^0(\gamma x;\xi).$$

$$\delta_\Gamma := dim_{Haus}(\Lambda_\Gamma),$$

where Λ_Γ is the limit set of Γ.

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Theorem (Guillarmou-Naud 2011)

Suppose that $\delta_\Gamma < (d-1)/2$. Then, for any $a \in C_c^\infty(T^*X)$, we have

$$\left\langle Op_h(a)E_h(\xi), E_h(\xi)\right\rangle_{L^2(X)} = \int_{S^*X} a\mathrm{d}\mu_\xi + O(h^{\min(1,d-2\delta)}).$$

Here, μ_ξ is supported on a fractal set of Hausdorff dimension $d+\delta_\Gamma$ containing Γ^+.

Hyperbolicity

Let X be a Euclidean or Hyperbolic near infinity manifold. We suppose the sectional curvature is everywhere nonpositive, and negative close to the trapped set.

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Let X be a Euclidean or Hyperbolic near infinity manifold. We suppose the sectional curvature is everywhere nonpositive, and negative close to the trapped set.
The dynamics is then *hyperbolic* close to K, so for each $\rho \in K$, we can define the unstable Jacobian as

$$J^{+}(\rho) := \frac{d}{dt}\Big|_{t=0} \det\left(d\Phi^{t}|_{E^{+}(\rho)}\right),$$

where $E^{+}(\rho)$ is the unstable space of ρ.

Topological pressure assumption

For each periodic orbit of p, we write T_p for its period, ρ_p for one of its points, and set $\tilde{J}^+(p) := \int_0^{T_p} J^+\big(\Phi^t(\rho_p)\big)\,dt$.
We then define the topological pressure associated to half the unstable Jacobian as

$$P(1/2) := \lim_{T\to\infty} \frac{1}{T} \log\Big(\sum_{T-1\leq T_p\leq T} \exp\big(\frac{-\tilde{J}^+(p)}{2}\big)\Big).$$

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In dimension 2, this is equivalent to saying that

$dim_{Haus}(K) < 2$, where dim_{Haus} is the Hausdorff dimension.

Statement of the results

Let X be Euclidean or Hyperbolic near infinity, with sectional curvature everywhere non-positive, and negative close to K. Suppose the topological pressure inequality holds.

Theorem (I.)

Let $\chi \in C_c^\infty(X)$. For any $r > 0$, $\ell > 0$, there exists $M_{r,\ell} > 0$ such that we have

$$\chi(x) E_h(x;\xi) = \sum_{n=0}^{\lfloor M_{r,\ell} |\log h| \rfloor} \sum_{j \in \mathcal{J}_n} e^{i\phi_{j,n}(x;\xi)/h} a_{j,n}(x;\xi,h) + R_r,$$

where $|\mathcal{J}_n|$ grows exponentially with n, and

$$\|R_r\|_{C^\ell} = O(h^r).$$

Main Property of the amplitudes

$$\chi(x)E_h(x;\xi) = \sum_{n=0}^{\lfloor M_{r,\ell}|\log h|\rfloor} \sum_{j\in\mathcal{J}_n} e^{i\phi_{j,n}(x;\xi)/h} a_{j,n}(x;\xi,h) + R_r$$

C^ℓ bounds

For any $\ell \in \mathbb{N}$, $\epsilon > 0$, there exists $C_{\ell,\epsilon}$ such that

$$\sum_{j\in\mathcal{J}_n} \|a_{j,n}\|_{C^\ell} \leq C_{\ell,\epsilon} e^{n(P(1/2)+\epsilon)}.$$

Corollaries(1)

Corollary (C^ℓ bounds)

For any $\ell \in \mathbb{N}$, we have

$$\|\chi E_h\|_{C^\ell} \leq C_\ell h^{-\ell}.$$

Main property of the phase

$$\chi(x)E_h(x;\xi) = \sum_{n=0}^{\lfloor M_{r,l}|\log h|\rfloor} \sum_{j\in\mathcal{J}_n} e^{i\phi_{j,n}(x;\xi)/h} a_{j,n}(x;\xi,h) + R_r$$

Distance between the Lagrangian leaves

$$|\partial\phi_{j,n}(x) - \partial\phi_{j',n'}(x)| > Ce^{b\min(n,n')},$$

where $b < 0$ is the minimal value taken by the sectional curvature on X.

Corollaries(2)

Corollary (Semiclassical measures)

For any $\epsilon > 0$*, for any* $a \in C_c^\infty(T^*X)$*, we have*

$$\langle Op_h(a)E_h(\xi), E_h(\xi)\rangle = \int_{T^*X} a(x,p)\mathrm{d}\mu_\xi(x,p) + O(h^{\min(1, \frac{|P(1/2)|}{2|b|} - \epsilon)}),$$

with

$$\mathrm{d}\mu_\xi(x,p) = \sum_{n=0}^{\infty}\sum_{j\in\mathcal{J}_n} |a^0_{j,n}(x;\xi)|^2 \delta_{\{p=\partial\phi_{j,n}(x;\xi)\}} dx,$$

where $a^0_{j,n}(x;\xi)$ *is the principal symbol of* $a_{j,n}(x;\xi)$*.*

Corollaries(3)

Let K be a compact subset of X. We define

$$\mathcal{N}_{h,K} := \{x \in K; \Re(E_h)(x) = 0\}.$$

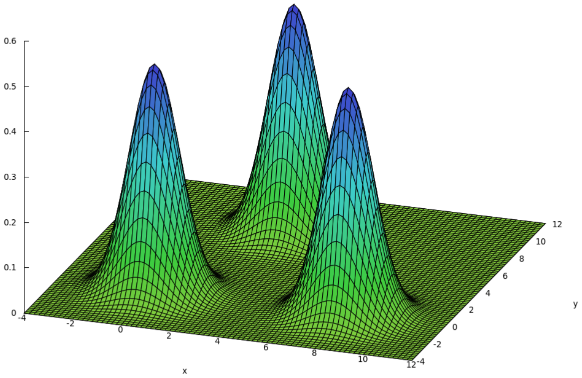
Corollary

There exists C_K such that

$$Haus_{d-1}\big(\mathcal{N}_{h,K}\big) \geq \frac{C_K}{h}.$$

More general setting

- "$h^2\Delta_g$" $\longrightarrow$ "$h^2\Delta_g + V$, $V \in C_c^\infty(X)$".
- "Sectional curvature everywhere nonpositive, and negative close to K" $\longrightarrow$ "K is a hyperbolic set for Φ^t".



More general setting

- Hyperbolicity close to the trapped set.
- Topological pressure assumption.
- Transversality assumption.

More general result

If Π_a is microlocalised close enough to a point in the trapped set, we have:

$$\mathcal{U}_a \Pi_a E_h(x) = \sum_{n=0}^{M_{r,l}|\log h|} \sum_{j \in \mathcal{J}_n} e^{i\phi_{j,n}(x)/h} a_{j,n}(x) + R_r,$$

where $\mathcal{U}_a$ is a FIO quantizing the use of adapted coordinates.

Sketch of proof

- Formally, we have

$$e^{-it/h}U(t)(\chi_0 E_h^0 + E_h^1) = E_h,$$

where $U(t) = e^{ith\Delta_g}$.

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- Resolvent estimates + "Hyperbolic Dispersion Estimates" $\Longrightarrow U(t)E_h^1$ becomes small as $t \longrightarrow \infty$.
- We have to study $U(t)\chi_0 E_h^0$ for long times. We want to use the WKB method.

Decomposing the phase space

We introduce $(W_a)_{a\in A}$ a finite open cover of S^*X in T^*X with

- $W_0 = S^*(X\backslash X_0)$ (no-interaction region)
- Some of the W_a are small sets close to K
- The others form an "intermediate region".

For any W_a close to the trapped set, we can equip it with an "adapted" system of symplectic coordinates. They are centred on $\rho_a \in K \cap W_a$, with axes tangent to the stable and unstable directions of the dynamics.

Evolution of Lagrangian manifolds

Incoming Lagrangian manifold:

$$\Lambda_\xi := \{(x,\xi), x \notin X_0\}.$$

We need to understand

$$\Lambda_\alpha := W_{\alpha_N} \cap \Phi^1(...\Phi^1(W_{\alpha_2} \cap \Phi^1(W_{\alpha_1} \cap \Lambda_\xi))...)$$

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From the "Inclination lemma", we can show that it is a finite union of a bounded number of Lagrangian manifolds, close to the unstable manifold.

Decomposition of the propagator

We take $(\Pi_a)_{a\in A}$ a quantum partition of unity on T^*X, with Π_a micro-supported in W_a.
If $\alpha \in A^N$, define

$$U_\alpha := \Pi_{\alpha_N} U(1) \Pi_{\alpha_{N-1}} U(1) ... U(1) \Pi_{\alpha_1}.$$

$$U(N) = \sum_{\alpha \in A^N} U_\alpha + O(h^\infty).$$

Each $U_\alpha \chi_0 E_h^0$ is a Lagrangian state, and we can estimate the norm of its symbol thanks to hyperbolicity and topological pressure.

Back to the decomposition

$$\chi(x)E_h(x) = \chi \sum_{\alpha \in A^N} U_\alpha \chi_0 E_h^0 + O(h^{r_N})$$
$$= \sum_{n=0}^{M_{r,l}|\log h|} \sum_{j \in \mathcal{J}_n} e^{i\phi_{j,n}(x)/h} a_{j,n}(x) + R_r.$$

n is the time spent inside the interaction region, and each $e^{i\phi_{j,n}(x)/h} a_{j,n}(x)$ corresponds to one of the Lagrangian states composing $U_\alpha \chi_0 E_h^0$.

Proof of Corollary 3

We use the Dong-Sogge-Zelditch formula: for any $f \in C_c^\infty(X)$,

$$\int_X \big((-h^2\Delta + 1)f\big)|\Re(E_h)|dV = h^2 \int_{\mathcal{N}_h} f|\nabla\Re(E_h)|dS.$$

Therefore,

$$\int_K |\Re(E_h)| \approx h^2 \int_{\mathcal{N}_{h,K}} |\nabla\Re(E_h)|dS \leq h^2\|\nabla E_h\|_{L^\infty} Haus_{d-1}(\mathcal{N}_{h,K}).$$

To bound the left-hand side, we use an equidistribution property.

Thank you for your attention !