Spectral gaps and resonance counting for hyperbolic manifolds

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Overview

Resonances: complex characteristic frequencies
describing exponential decay of waves in open systems

$\operatorname{Re}\lambda_j$ = rate of oscillation, $-\operatorname{Im}\lambda_j$ = rate of decay

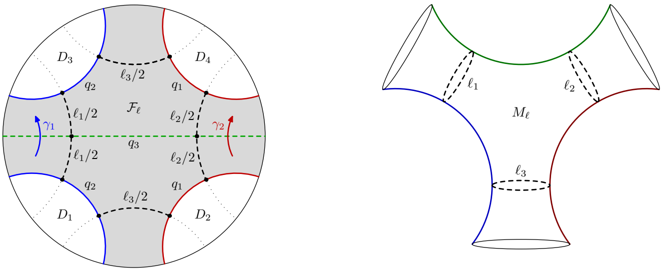
Our setting: convex co-compact hyperbolic manifolds

The high frequency régime $|\operatorname{Im}\lambda| \leq C$, $|\operatorname{Re}\lambda| \gg 1$
is governed by the set of trapped trajectories,
which in our case is determined by the limit set Λ_Γ

We give a new spectral gap and fractal Weyl bound
for resonances using a "fractal uncertainty principle"

Hyperbolic manifolds

$(M, g) = \Gamma \backslash \mathbb{H}^n$ convex co-compact hyperbolic manifold



An example: three-funnel surface with neck lengths ℓ_1, ℓ_2, ℓ_3

Resonances: poles of the scattering resolvent

$$R(\lambda) = \Big(-\Delta_g - \frac{(n-1)^2}{4} - \lambda^2 \Big)^{-1} : \begin{cases} L^2(M) \to L^2(M), & \operatorname{Im}\lambda > 0 \\ L^2_{\mathrm{comp}}(M) \to L^2_{\mathrm{loc}}(M), & \operatorname{Im}\lambda \le 0 \end{cases}$$

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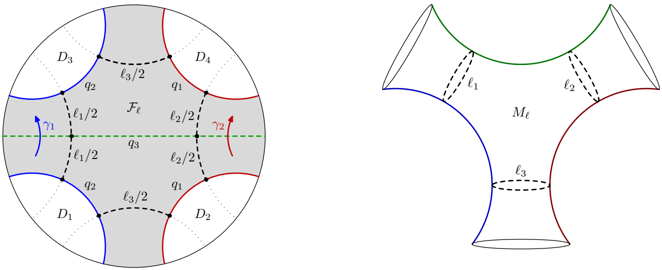

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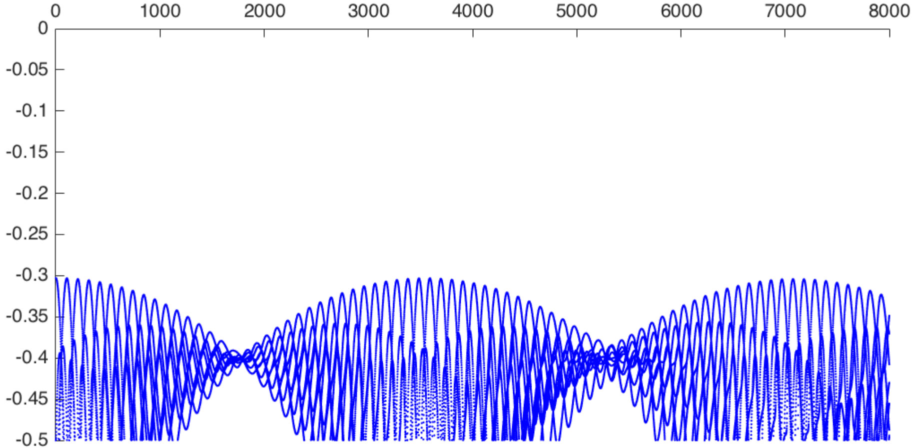
Resonances: poles of the scattering resolvent

Also correspond to poles of the Selberg zeta function

Existence of meromorphic continuation: Patterson '75,'76, Perry '87,'89, Mazzeo–Melrose '87, Guillopé–Zworski '95, Guillarmou '05, Vasy '13

Plots of resonances

Three-funnel surface with $\ell_1 = \ell_2 = \ell_3 = 7$

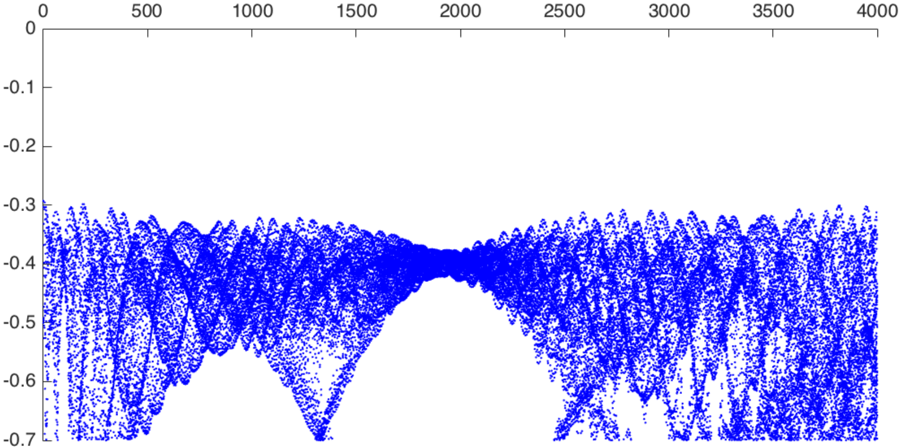


Data courtesy of David Borthwick and Tobias Weich

See arXiv:1305.4850 and arXiv:1407.6134 for more

Plots of resonances

Three-funnel surface with $\ell_1 = 6$, $\ell_2 = \ell_3 = 7$



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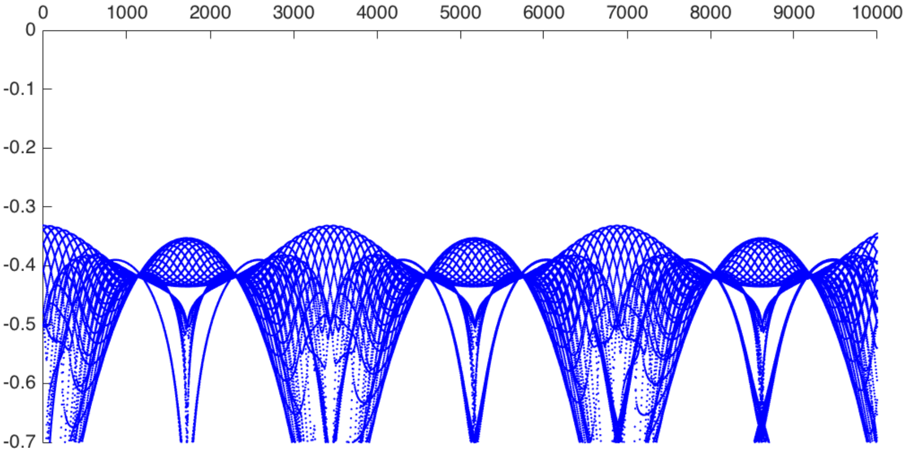
Plots of resonances

Torus-funnel surface with $\ell_1 = \ell_2 = 7$, $\varphi = \pi/2$, trivial representation



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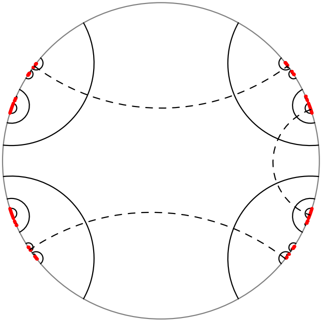
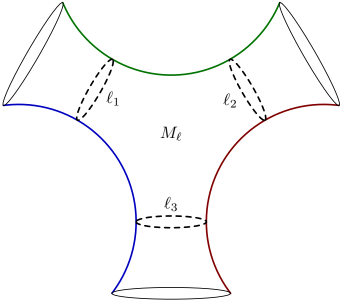
The limit set and δ

$M = \Gamma\backslash\mathbb{H}^n$ hyperbolic manifold

$\Lambda_\Gamma \subset \mathbb{S}^{n-1} = \partial\overline{\mathbb{H}^n}$ the limit set

$\delta := \dim_H(\Lambda_\Gamma) \in [0, n-1]$





Trapped geodesics: both endpoints in Λ_Γ

Forward/backward trapped: one endpoint in Λ_Γ

Essential spectral gap

Essential spectral gap of size $\beta > 0$:

only finitely many resonances with $\operatorname{Im}\lambda > -\beta$

One application: resonance expansions of waves with $\mathcal{O}(e^{-\beta t})$ remainder

Patterson–Sullivan: the topmost resonance is $\lambda = i(\delta - \frac{n-1}{2})$, therefore there is a gap of size $\beta = \max\left(0, \frac{n-1}{2} - \delta\right)$

See also Ikawa '88, Gaspard–Rice '89, Nonnenmacher–Zworski '09

$\delta > \frac{n-1}{2}$ $\delta < \frac{n-1}{2}$

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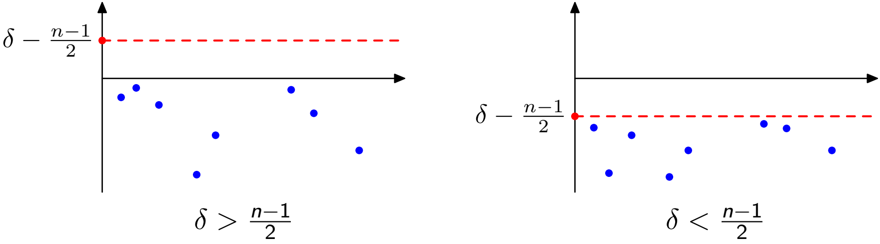


Asymptotic spectral gap: finitely many resonances with $\operatorname{Im}\lambda > -\beta$
Standard gap: $\beta_{\mathrm{std}} = \max(0, \frac{n-1}{2} - \delta)$

Naud '04, Stoyanov '11 (using Dolgopyat '98):
gap of of size $\frac{n-1}{2} - \delta + \varepsilon$ for $0 < \delta \leq \frac{n-1}{2}$ and $\varepsilon > 0$ depending on M

Theorem 1 [D–Zahl '15]

There is a gap of size

$$\beta = \frac{3}{8}\Big(\frac{n-1}{2} - \delta\Big) + \frac{\beta_E}{16}$$

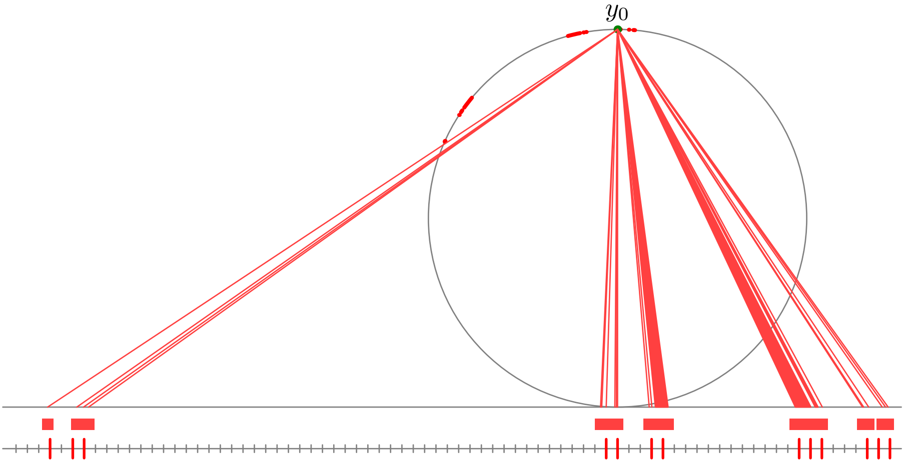
where $\beta_E \in [0, \delta]$ is the improvement in the asymptotic of additive energy of the limit set. For surfaces, we furthermore have

$$\beta_E > \delta \exp\big[-K(1-\delta)^{-28}\log^{14}(1+C)\big]$$

where C is the constant in the δ-regularity of the limit set and K is a global constant. This improves over β_{std} for $\delta = \frac{1}{2}$ and nearby surfaces, including some with $\delta > \frac{1}{2}$

Additive energy

$X(y_0, \alpha) \subset (\alpha\mathbb{Z} \cap [-1,1])^{n-1}$ discretization of Λ_Γ projected from $y_0 \in \Lambda_\Gamma$



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Additive energy:

$$E_A(y_0,\alpha) = \#\{(a,b,c,d) \in X(y_0,\alpha)^4 \mid a+b=c+d\}$$
$$|X(y_0,\alpha)| \sim \alpha^{-\delta}, \quad \alpha^{-2\delta} \lesssim E_A(y_0,\alpha) \lesssim \alpha^{-3\delta}$$

Definition

Λ_Γ has improved additive energy with exponent $\beta_E \in [0,\delta]$, if

$$E_A(y_0,\alpha) \le C\alpha^{-3\delta+\beta_E}, \quad 0<\alpha<1,$$

where C does not depend on y_0

Random sets have improved additive energy with $\beta_E = \min(\delta, n-1-\delta)$

Theorem [D–Zahl '15]

For convex co-compact hyperbolic surfaces, there is a gap of size

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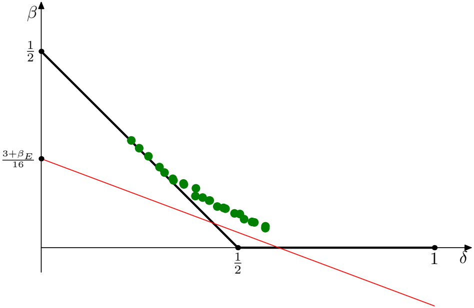
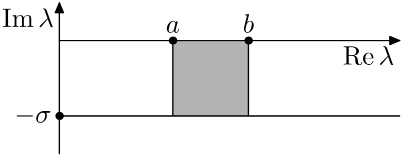


Numerics for 3- and 4-funneled surfaces by Borthwick–Weich '14
+ our gap for $\beta_E := \delta$ (representing some wishful thinking)

Counting resonances

Denote by $N_{[a,b]}(\sigma)$ the number of resonances with

$$\operatorname{Re}\lambda \in [a, b], \quad \operatorname{Im}\lambda \geq -\sigma$$



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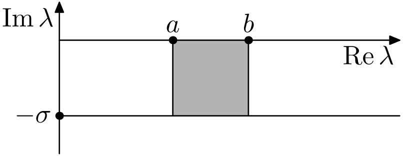

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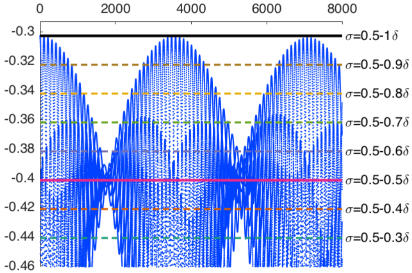

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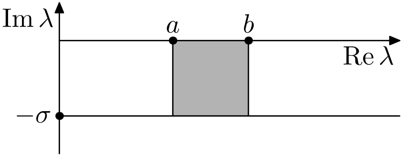

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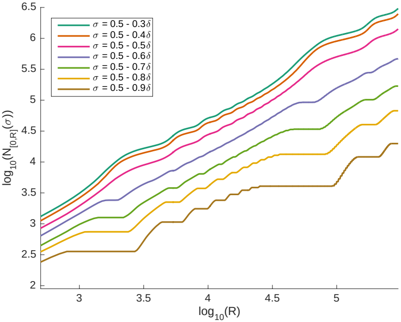

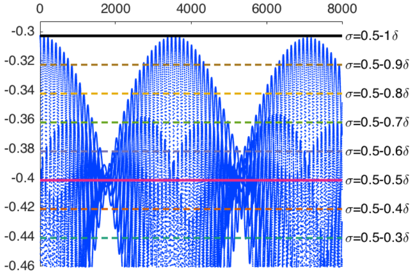

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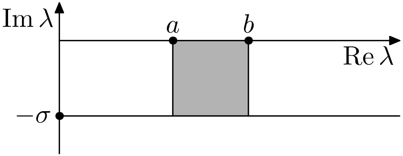

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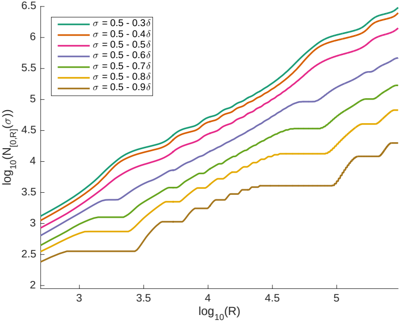
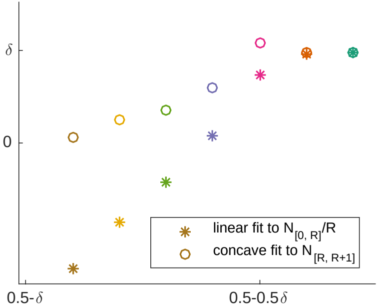




Fractal Weyl bounds

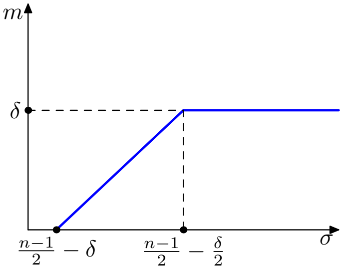
$N_{[a,b]}(\sigma) = \#\{\text{resonances with } \operatorname{Re}\lambda \in [a,b],\ \operatorname{Im}\lambda > -\sigma\}$

Theorem 2 [D '15]

For σ fixed and $R \to \infty$, $N_{[R,R+1]}(\sigma) = \mathcal{O}(R^{m(\sigma,\delta)+})$, where

$$m(\sigma,\delta) = \min(2\delta + 2\sigma - (n-1), \delta).$$

Note that $m = 0$ at $\sigma = \frac{n-1}{2} - \delta$ and $m = \delta$ starting from $\sigma = \frac{n-1}{2} - \frac{\delta}{2}$



Zworski '99, Guillopé–Lin–Zworski '04, Datchev–D '13: $N_{[R,R+1]}(\sigma) = \mathcal{O}(R^{\delta})$

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Naud '14, Jakobson–Naud '14: For $n = 2$, $N_{[0,R]}(\sigma) = \mathcal{O}(R^{1+\gamma})$, for some $\gamma(\sigma, M) < \delta$ when $\sigma < \frac{1}{2} - \frac{\delta}{2}$

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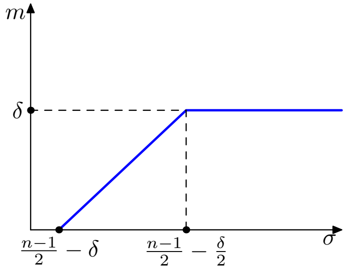

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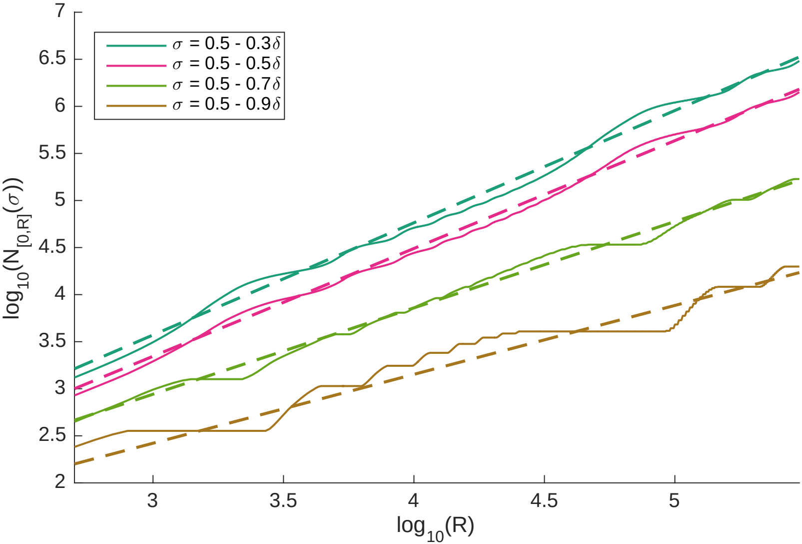
Fractal Weyl bounds in pictures



The dashed lines are least squares linear fits to $\log_{10} N_{[0,R]}(\sigma)$

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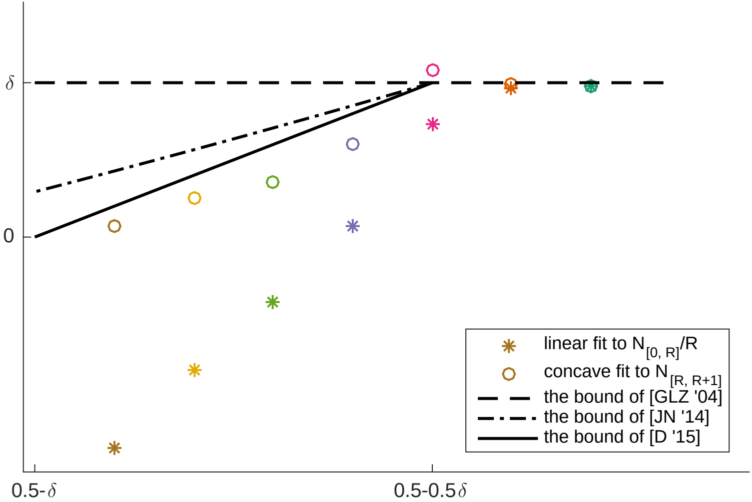
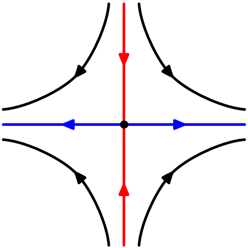


A comparison of numeric fits with the bounds of Guillopé–Lin–Zworski '04, Jakobson–Naud '14, and D '15

Dynamics of the geodesic flow



$M = \Gamma\backslash\mathbb{H}^n$ convex co-compact hyperbolic manifold

The homogeneous geodesic flow

$$\varphi^t : T^*M \setminus 0 \to T^*M \setminus 0$$

is hyperbolic with weak (un)stable foliations L_u/L_s

Incoming/outgoing tails:

$$\Gamma_+ = \{(x,\xi) \mid \varphi^t(x,\xi) \not\to \infty \text{ as } t \to -\infty\}$$
$$\Gamma_- = \{(x,\xi) \mid \varphi^t(x,\xi) \not\to \infty \text{ as } t \to +\infty\}$$

On the cover $T^*\mathbb{H}^n \setminus 0$,
Γ_+/Γ_- are foliated by L_u/L_s and look similar to the limit set Λ_Γ in directions transversal to L_u/L_s

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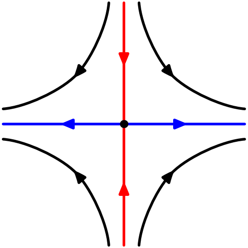

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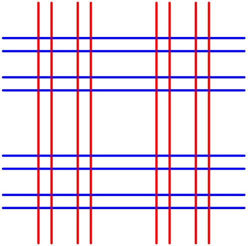

Microlocalization of resonant states

Assume $\lambda = h^{-1} - i\nu$ is a resonance, $0 < h \ll 1$. There is a resonant state

$$\Big(-\Delta_g - \frac{(n-1)^2}{4} - \lambda^2\Big)u = 0,\quad u \text{ outgoing at infinity},\quad \|u\| = 1$$

Vasy '13: extend u to an eigenstate of a Fredholm problem on $M_{\mathrm{ext}} \supset \overline{M}$

Microlocally, u lives near Γ_+, has positive mass on Γ_-, and

$$u = e^{i\lambda t}U(t)u;\quad U(t) = e^{-it\sqrt{-\Delta_g-(n-1)^2/4}} \text{ quantizes } \varphi^t$$

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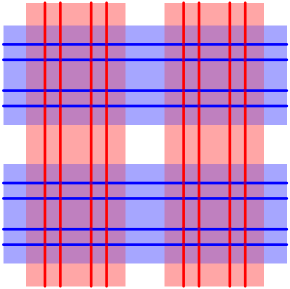
Outgoing condition implies:

$$u = \mathrm{Op}_h(\chi_+) u + \mathcal{O}(h^\infty),$$
$$\|\mathrm{Op}_h(\chi_-) u\| \geq C^{-1}$$
$$\operatorname{supp} \chi_\pm \subset \ \varepsilon\text{-neighborhood of } \Gamma_\pm$$



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Propagation for time t:

$$u = \mathrm{Op}_h(\chi_+) u + \mathcal{O}(h^\infty),$$
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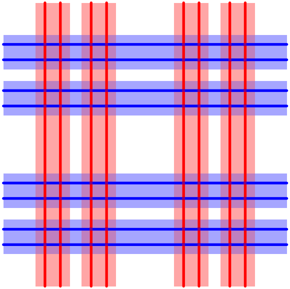

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Propagation for time $t = \log(1/h)$:

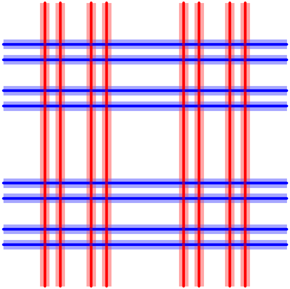
$$u = \mathrm{Op}_h^{L_u}(\chi_+) u + \mathcal{O}(h^\infty),$$

$$\|\mathrm{Op}_h^{L_s}(\chi_-) u\| \geq C^{-1} e^{-\nu t} = C^{-1} h^\nu$$

$$\mathrm{supp}\, \chi_\pm \subset \ h\text{-neighborhood of } \Gamma_\pm$$

Use second microlocal calculi associated to L_u / L_s

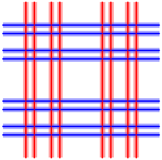
In practice, we take $t = \rho \log(1/h)$, $\rho = 1 - \varepsilon$



u a resonant state at $\lambda = h^{-1} - i\nu$, $\quad \|u\| = 1$

$u = \mathrm{Op}_h^{L_u}(\chi_+)u + \mathcal{O}(h^\infty)$, $\quad \|\mathrm{Op}_h^{L_s}(\chi_-)u\| \geq C^{-1}h^\nu$

$\operatorname{supp}\chi_\pm \subset$ h-neighborhood of $\Gamma_\pm \cap S^*M$



Proof of Theorem 1 (gaps)

- To get a gap of size β, enough to show a fractal uncertainty principle:
$$\|\mathrm{Op}_h^{L_s}(\chi_-)\mathrm{Op}_h^{L_u}(\chi_+)\|_{L^2\to L^2} \ll h^\beta$$
- A basic bound gives the standard gap $\beta = \frac{n-1}{2} - \delta$:
$$\|\mathrm{Op}_h^{L_s}(\chi_-)\mathrm{Op}_h^{L_u}(\chi_+)\|_{\mathrm{HS}} \leq Ch^{\frac{n-1}{2}-\delta} \qquad (1)$$
- The bound via additive energy is obtained by harmonic analysis in L^4

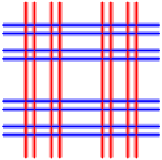
Proof of Theorem 2 (counting)

- First write for each resonant state, $u = \mathcal{A}(\lambda)u$,
$\mathcal{A}(\lambda) = Y(\lambda)\mathrm{Op}_h^{L_s}(\chi_-)\mathrm{Op}_h^{L_u}(\chi_+) + \mathcal{O}(h^\infty)$, $\quad \|Y(\lambda)\| \leq Ch^{-\nu}$
- Next estimate $\det(I - \mathcal{A}(\lambda)^2) \leq \exp(\|\mathcal{A}(\lambda)\|_{\mathrm{HS}}^2)$ using (1)

u a resonant state at $\lambda = h^{-1} - i\nu, \quad \|u\| = 1$

$$u = \mathrm{Op}_h^{L_u}(\chi_+)u + \mathcal{O}(h^\infty), \quad \|\mathrm{Op}_h^{L_s}(\chi_-)u\| \geq C^{-1}h^\nu$$

$\operatorname{supp}\chi_\pm \subset$ h-neighborhood of $\Gamma_\pm \cap S^*M$



Proof of Theorem 1 (gaps)

- To get a gap of size β, enough to show a fractal uncertainty principle:
$$\|\mathrm{Op}_h^{L_s}(\chi_-)\mathrm{Op}_h^{L_u}(\chi_+)\|_{L^2\to L^2} \ll h^\beta$$
- A basic bound gives the standard gap $\beta = \frac{n-1}{2} - \delta$:
$$\|\mathrm{Op}_h^{L_s}(\chi_-)\mathrm{Op}_h^{L_u}(\chi_+)\|_{\mathrm{HS}} \leq Ch^{\frac{n-1}{2}-\delta} \tag{1}$$
- The bound via additive energy is obtained by harmonic analysis in L^4

Proof of Theorem 2 (counting)

- First write for each resonant state, $u = \mathcal{A}(\lambda)u$,
$\mathcal{A}(\lambda) = Y(\lambda)\mathrm{Op}_h^{L_s}(\chi_-)\mathrm{Op}_h^{L_u}(\chi_+) + \mathcal{O}(h^\infty), \quad \|Y(\lambda)\| \leq Ch^{-\nu}$
- Next estimate $\det(I - \mathcal{A}(\lambda)^2) \leq \exp(\|\mathcal{A}(\lambda)\|_{\mathrm{HS}}^2)$ using (1)

Thank you for your attention!