# Expanders and box spaces

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# 1 The objects

Expanders are remarkably ubiquitous objects, both in mathematics and computer science.

# Expanders Everywhere!

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Expanders are families of sparse but highly connected graphs.

**Definition 1.1.** *A family $(X_n)_{n>0}$ of finite, connected, d-regular graphs is an expander if the following equivalent conditions hold:*

- *The isoperimetric, or Cheeger constant $h(X_n)$ is bounded below by a positive constant: $h(X_n) \geq \epsilon$ for every $n$.*
- *The first eigenvalue $\lambda_1(X_n)$ of the Laplace operator on $X_n$, is bounded below by a positive constant: $\lambda_1(X_n) \geq \epsilon'$ for every $n$.*

We turn a family $(X_n)_{n>0}$ of finite connected graphs (with unbounded diameter) into a metric space via the *coarse disjoint union*: on $X = \coprod_n X_n$, put a metric $d$ such that $d|_{X_n}$ is the graph metric, and

$$d(X_m, X_n) > \max\{diam(X_m), diam(X_n)\}$$

for $m \neq n$.

If $G$ is a finitely generated, residually finite group, a *filtration* of $G$ is a decreasing sequence $(N_i)_{i>0}$ of finite index normal subgroups with $\cap_i N_i = \{1\}$. Let $S$ be a finite, symmetric, generating set of $G$.

**Definition 1.2.** *The box space of $G$ (w.r.t. the filtration $(N_i)_{i>0}$) is the coarse disjoint union of the Cayley graphs $\Box_{(N_i)} G = \coprod_i Cay(G/N_i, S)$.*

To get something more canonical, we occasionally consider the *full box space*

$$\Box_f G = \coprod_{N \triangleleft G, [G:N]<\infty} Cay(G/N, S).$$

Up to coarse equivalence, box spaces do *not* depend on $S$.

Recall the first explicit construction of expanders by Margulis in 1973:

**Theorem 1.3.** *Any box space of a countable, residually finite group with Kazhdan's property (T), is an expander.*

**Example 1.** *Let* $\Gamma(N) = \ker[SL_d(\mathbf{Z}) \to SL_d(\mathbf{Z}/N\mathbf{Z})]$ *be a congruence subgroup in* $SL_d(\mathbf{Z})$*. For* $d \geq 3$ *and a prime* $p$*, the box space* $\Box_{\Gamma(p^n)} SL_d(\mathbf{Z})$ *is an expander.*

## 2 Around the coarse Baum-Connes conjecture

The *coarse Baum-Connes conjecture* says that, for a discrete metric space with bounded geometry, the "analytical index" map

$$\mu_X : K_*^{top}(X) \to K_*(C^*(X))$$

is an isomorphism. Here $K_*^{top}(X)$ is a topological gadget (containing large scale, topological algebraic nature) while $K_*(C^*(X))$ is an analytic/algebraic gadget.

It's not a conjecture anymore!

**Theorem 2.1.** *(N. Higson, V. Lafforgue, G. Skandalis 2002) Let $X$ be any congruence box space of $SL_d(\mathbf{Z})$ $(d \geq 2)$. The map $\mu_X$ is not surjective.*

This seminal result (80 references according to MathSciNet!) launched a wealth of activity:

- The *maximal* coarse Baum-Connes conjecture holds for certain expanders (H. Oyono-Oyono, G. Yu 2009)
- For expanders with large girth, the analytical index map is injective but not surjective (R. Willett, G. Yu 2010)
- The maximal coarse Baum-Connes conjecture holds for families of graphs with girth tending to infinity (R. Willett, G. Yu 2012)
- Introduction by P. Baum, E. Guentner and R. Willett (2013) of the *exact crossed product* to reformulate the Baum-Connes conjecture: all confirming examples remain confirming examples, and some previous expander-based counterexamples now become confirming examples.

## 3 Coarse geometry

Let $(X, d_X), (Y, d_Y)$ be unbounded metric spaces.

**Definition 3.1.** *A map $f : X \to Y$ is a coarse embedding if, for every two sequences $(x_n)_{n>0}, (y_n)_{n>0}$ in $X$:*

$$d_X(x_n, y_n) \to +\infty \iff d_Y(f(x_n), f(y_n)) \to +\infty$$

With

$$\rho_-(t) =: \inf\{d_Y(f(x), f(y)) : d_X(x, y) \geq t\};$$

$$\rho_+(t) =: \sup\{d_Y(f(x), f(y) : d_X(x, y) \leq t\},$$

we get the equivalent definition

**Definition 3.2.** *$f$ is a coarse embedding if there exists control functions $\rho_\pm : \mathbf{R}^+ \to \mathbf{R}$, with $\lim_{t\to\infty} \rho_\pm(t) = +\infty$, such that for every $x, y \in X$:*

$$\rho_-(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \rho_+(d_X(x, y)).$$

**Definition 3.3.** *A coarse embedding $f : X \to Y$ is a coarse equivalence if $f$ is quasi-surjective, i.e. there exists $R > 0$ such that $Y$ is the $R$-neighborhood of $f(X)$.*

A dictionary between coarse-geometry properties of $\square G$ and group-theoretical properties of $G$:

The property (T) side:

| $\square G$ | $G$ |
|---|---|
| $\square_{(N_k)} G$ has geometric property (T) | $G$ has property (T) (Willett & Yu 2013) |
| $\square_{(N_k)} G$ is an expander family | $G$ has property $(\tau)$ w.r.t. the filtration $(N_k)_{k>0}$ (Lubotzky-Zimmer 1989) |

Turning to the amenability side:

| $\Box G$ | $G$ |
|---|---|
| $\Box_{(N_k)}G$ has linear diameter | $G$ is virtually cyclic (Khukhro & V. 2015) |
| $\Box_{(N_k)}G$ has Yu's property A | $G$ is amenable (Roe 2003) |
| $\Box_{(N_k)}G$ admits a fibered coarse embedding into $L^2$ | $G$ is a-(T)-menable (Chen-Wang-Wang 2013; Finn-Sell 2013) |

Note that $SL_2(\mathbf{Z})$ is a-(T)-menable but has property $(\tau)$ with respect to congruence subgroups. If $p$ is a prime, $SL_2(\mathbf{Z}[\sqrt{p}])$ is a-(T)-menable but has property $(\tau)$ (w.r.t all finite index subgroups).

## 4 How many expanders?

**Theorem 4.1.** *(M. Mendel and A. Naor 2011) There exists an expander $X$ with unbounded girth, and $Y$ an expander with many short cycles, such that $Y$ does not coarsely embed in $X$.*

Geometric property (T) (R. Willett and G. Yu 2013) distinguishes expanders from property (T) groups, from expanders from a-(T)-menable groups.

**Theorem 4.2.** *(D. Hume 2014) There exists a continuum of pairwise coarsely inequivalent expanders with unbounded girth.*

This is proved using *separation profile* of Benjamini-Schramm-Timar (2010).

**Theorem 4.3.** *(A. Khukhro-V. 2015)*

1) *For $d \geq 2$: the group $SL_d(\mathbf{Z})$ admits a continuum of pairwise inequivalent box spaces which are expanders (in particular, for $d \geq 3$: there exists a continuum of expanders with geometric property (T)).*

2) *For $m \neq n$: a box space of $SL_m(\mathbf{Z})$ is not coarsely equivalent to a box space of $SL_n(\mathbf{Z})$.*

3) *For $p, q$ distinct primes: a box space of $SL_2(\mathbf{Z}[\sqrt{p}]$ is not coarsely equivalent to a box space of $SL_2(\mathbf{Z}[\sqrt{q}])$.*

As Willett and Yu proved that a space with geometric property (T) does not coarsely embed in a space admitting a fibered coarse embedding into Hilbert space, we get:

**Corollary 4.4.** *A box space of $SL_d(\mathbf{Z})$ (for $d \geq 3$) does not coarsely embed into a box space of $SL_2(\mathbf{Z}[\sqrt{p}])$.*

## 5 On the proofs

Let $X = \coprod_n X_n$ and $Y = \coprod_n Y_n$ be two coarse disjoint unions of finite, connected graphs with degrees bounded by $d$, with $diam|X_n|, diam|Y_n| \to \infty$.

**Lemma 5.1.** *If $f : X \to Y$ is a coarse equivalence, then:*

1. *There exists $A \geq 1$ such that for every $n \geq 1$ and $x, y \in X_n$:*

   $$\frac{1}{A} d_{X_n}(x,y) - A \leq d_Y(f(x), f(y)) \leq A d_{X_n}(x,y) + A$$

   *($f$ is a family of quasi-isometries with uniform constants)*

2. *$f$ induces a bijection $\alpha$ between a co-finite set of components of $X$ and a co-finite set of components of $Y$ (say that $\alpha$ is an almost bijection).*

3. *There exists $C \geq 1$ such that, for $n \gg 0$:*

   $$\frac{|X_n|}{C} \leq |Y_{\alpha(n)}| \leq C.|X_n|$$

**Proof:**

1. For $x, y \in X_n$, we get by the triangle inequality: $d_Y(f(x), f(y)) \leq \rho_+(1).d_{X_n}(x, y)$. Using a quasi-inverse for $f$, get the converse inequality.

2. For $n \gg 0$ we have $f(X_n) \subset Y_m$. Set $m = \alpha(n)$.

   For $n, n' \gg 0$: $X_n$, $X_{n'}$ are far away, so not mapped to the same component: $\alpha(n) \neq \alpha(n')$.

   If $Y$ is the $R$-neighborhood of $f(X)$: for $m \gg 0$, we have $diam(Y_m) \geq R$, so $Y_m$ meets $f(X)$, i.e. $m = \alpha(n)$ for some $n$.

3. Since $f$ has uniformly bounded fibers: $\frac{|X_n|}{K} \leq |f(X_n)| \leq |Y_{\alpha(n)}|$. On the other hand $|Y_{\alpha(n)}| \leq |B_{T_d}(R)|.|f(X_n)| \leq K'.|X_n|$. $\square$

**Definition 5.2.** *A filtration $(N_i)_{i>0}$ of $G$ is strict, if the sequence $(N_i)$ is strictly decreasing.*

**Lemma 5.3.** *Let $(M_i)_{i>0}, (N_i)_{i>0}$ be strict filtrations of $G, H$ respectively. Let $f : \Box_{(M_i)} G \to \Box_{(N_i)} H$ be a coarse equivalence. Then the almost bijection $\alpha$ has bounded displacement, i.e. there is $N \in \mathbf{N}$ such that $|\alpha(n) - n| \leq N$ for $n \gg 0$.*

**Sketch of Proof:** Set $X_i = G/M_i, Y_i = H/N_i$. By previous lemma: $|\log_2 |X_i| - \log_2 |Y_{\alpha(i)}|| \leq K$. Now $|X_j| \geq 2^{j-i}|X_i|$ for $j \geq i$, hence $\log_2 |X_j| - \log_2 |X_i| \geq j - i$. Take $N > 2K$, to ensure $\log_2 |Y_{\alpha(i)}| < \log_2 |Y_{\alpha(j)}|$ if $i + N \leq j$, i.e. $\alpha(i) < \alpha(j)$ if $i + N \leq j$. From this it follows that $|\alpha(i) - i| \leq N$. $\Box$

Hence necessary condition for coarse equivalence:

**Proposition 5.4.** *If box spaces $\Box_{(M_i)} G, \Box_{(N_i)} H$ (associated with strict filtrations) are coarsely equivalent, then for some almost bijection $\alpha$ the ratios $\frac{|G/M_i|}{|H/N_{\alpha(i)}|}$ and $\frac{|H/N_{\alpha(i)}|}{|G/M_i|}$ are bounded.*
$\Box$

Now we deal with congruence box spaces of $SL_d(\mathbf{Z})$. For $p$ prime:

$$|SL_d(\mathbf{Z}/p^k\mathbf{Z})| = p^{k(d^2-1)}(1-\frac{1}{p^d})(1-\frac{1}{p^{d-1}})...(1-\frac{1}{p^2})$$

From this we deduce:

**Proposition 5.5.**

- *For $p, q$ primes, $m, n \geq 2$, the coarse disjoint unions $\coprod_k SL_m(\mathbf{Z}/p^k\mathbf{Z})$ and $\coprod_k SL_n(\mathbf{Z}/q^k\mathbf{Z})$ (viewed as box spaces of $SL_m(\mathbf{Z}), SL_n(\mathbf{Z})$ respectively) are coarsely equivalent if and only if $m = n$ and $p = q$.*
- *For $s \geq 1$, set $N_k(s) = 2^{[ks]}$. The expanders $\coprod_k SL_m(\mathbf{Z}/N_k(s)\mathbf{Z})$ (viewed as box spaces of $SL_m(\mathbf{Z})$) are pairwise coarsely inequivalent for $s \geq 1$.* □

## 6 Quasi-isometry of groups

**Theorem 6.1.** *(suggested by R. Tessera) If box spaces* $\square_{(M_i)}G$, $\square_{(H_i)}H$ *are coarsely equivalent, then* $G$ *and* $H$ *are quasi-isometric.*

R. Tessera has announced that his PhD student K. Das has a stronger result: under the same assumption, $G$ and $H$ are uniformly measure equivalent (in particular $\beta_i^{(2)}(G) = 0$ iff $\beta_i^{(2)}(H) = 0$).

The Theorem above leads to the subject of quasi-isometric rigidity, where many results are available:

- The quasi-isometry class of $SL_2(\mathbf{Z}[\sqrt{p}])$ remembers $p$ (B. Farb and R. Schwartz 1996);
- The quasi-isometry class of $SL_m(\mathbf{Z})$ remembers $m$ (A. Eskin 1998).

**Idea of proof:** Let $f : \Box_{(M_i)}G \to \Box_{(N_i)}H$ be a coarse equivalence; so there exists $A \geq 1$ such that, for $i \gg 0$, $f$ is a $(A, A)$-quasi-isometry $G/M_i \to H/N_{\alpha(i)}$. Fix $N \geq 0$. The restriction $f|_{G/M_i}$ maps the ball $B_i(N)$ around 1 in $G/M_i$, to the ball $B_{\alpha(i)}((A+1)N)$ around 1 in $H/N_{\alpha(i)}$. For $i \gg 0$, by residual finiteness, the ball $B_G(N)$ around 1 in $G$ is isometric to $B_i(N)$, and the ball $B_H((A+1)N)$ in $H$ is isometric to $B_{\alpha(i)}((A+1)N)$. So we get a family of maps $f_{N,i} : B_G(N) \to B_H((A+1)N)$ (one for every $i \gg 0$). There are finitely many maps $B_G(N) \to B_H((A+1)N)$. Using a diagonal argument, we may extract from all these maps a $(A, A)$-quasi-isometry $G \to H$. $\Box$

## 7 Large diameter

**Definition 7.1.** *Fix* $\alpha \in ]0,1]$. *A sequence* $(X_n)_{n>0}$ *of finite, connected, d-regular graphs, with* $|X_n| \to \infty$, *satisfies property* $(D_\alpha)$ *if for some constant* $C > 0$ *we have*

$$diam(X_n) \geq C.|X_n|^\alpha.$$

Recall that, for an expander, the diameter of $X_n$ is logarithmic in $|X_n|$, so property $(D_\alpha)$ is a strong form of non-expansion.

**Theorem 7.2.** *(A.K & A.V. 2015) For G residually finite, finitely generated:* $\square_{(N_i)} G$ *has* $(D_1)$ *if and only if* $G$ *is virtually cyclic.*

The proof uses the pre-Gromov result: a group with linear growth is virtually cyclic.

**Theorem 7.3.** *((1) $\Rightarrow$ (2): E. Breuillard & M. Tointon 2015; (2) $\Rightarrow$ (1): A.K& A.V., 2015) For $G$ residually finite and finitely generated, TFAE:*

1. *Some box space of $G$ has $(D_\alpha)$, for some $\alpha \in ]0,1]$.*
2. *$G$ virtually maps onto $\mathbf{Z}$.*

**Proof** when $G$ maps onto $\mathbf{Z}$, i.e. $G = H \rtimes \mathbf{Z}$. Then any $\alpha < 1$ does the job.
Write $\mathbf{Z} =< t >$. Let $(M_n)_{n>0}$ be a filtration of $G$. Let $k_n$ be a sequence of integers such that $k_n$ divides $k_{n+1}$ and $k_n$ is a multiple of the order of $Ad(t)$ on $H/(H \cap M_n)$. Then $N_n =< H \cap M_n, t^{k_n} >$ is a filtration of $G$ and

$$diam(G/N_n) \geq diam(C_{k_n}) = \frac{k_n}{2}.$$

So by assuming

$$k_n \geq |H \cap M_n|^{\frac{\alpha}{1-\alpha}},$$

we get $(D_\alpha)$. $\square$

**Corollary 7.4.** *Fix* $\alpha \in ]0,1[$. *If* $M$ *is a closed Riemannian manifold, and* $\pi_1(M)$ *virtually maps onto a non-amenable, residually finite group with infinite abelianization; then* $M$ *admits a tower* $(M_n)_{n>0}$ *of finite-sheeted coverings, with* $\lambda_1(M_n) = O(vol(M_n)^{-\alpha})$, *such that the coarse union of the* $M_n$*'s is not coarsely equivalent to any coarse union of finite-sheeted covers of* $M$ *obtained by (virtually) mapping* $\pi_1(M)$ *onto a residually finite amenable group.* □

This generalizes a result by G. Arzhantseva and E. Guentner (2012), under stronger assumption that $\pi_1(M)$ virtually maps onto $\mathbf{F}_2$.

HAPPY BIRTHDAY GEORGES,
MANY HAPPY RETURNS!