HAPPY BIRTHDAY

GEORGES !

Quanta of Geometry

Alain Connes

Joint work with **Ali Chamseddine**
Viatcheslav Mukhanov
and **Walter van Suijlekom**

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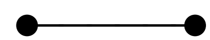
Geometry from the Quantum

The goal is to reconcile Quantum Mechanics and General Relativity by showing that the latter naturally arises from a higher degree version of the Heisenberg commutation relations.

with A. Chamseddine and S. Mukhanov

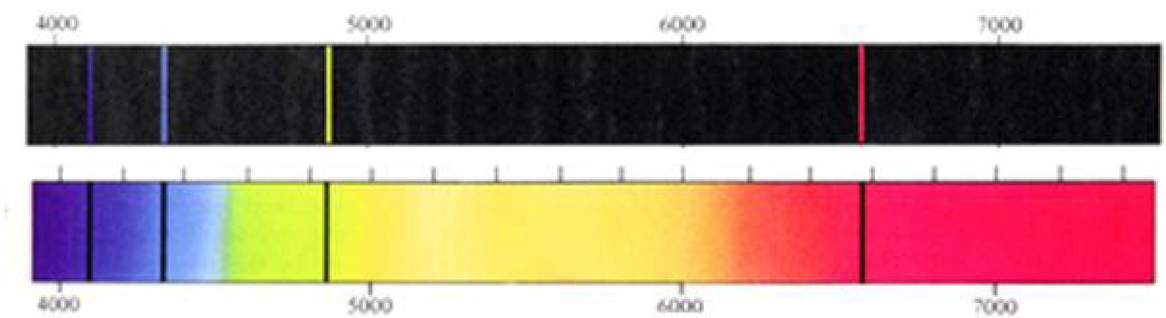
We have discovered a geometric analogue of the Heisenberg commutation relations $[p, q] = i\hbar$. The role of the momentum p is played by the Dirac operator. It plays the role of a measuring rod and at an intuitive level it represents the inverse of the line element ds familiar in Riemannian geometry

$$ds =$$ 

Spectral triples

$$(\mathcal{A}, \mathcal{H}, D)\,, \quad ds = D^{-1}\,,$$

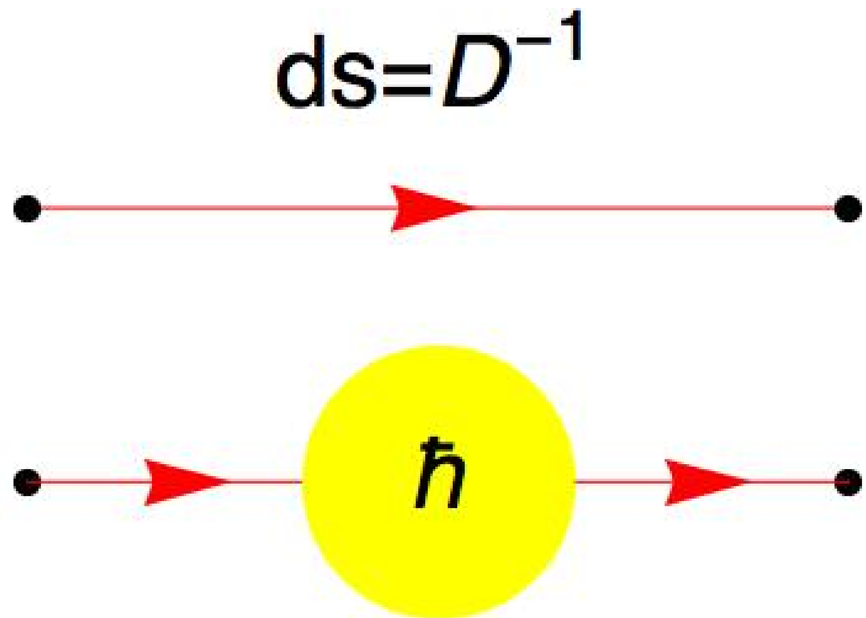
$$d(A,B) = \mathrm{Sup}\,\{|f(A) - f(B)|\,;\, f \in \mathcal{A}\,,\ \|[D,f]\| \leq 1\,\}$$



Meter $\to$ Wave length (Krypton (1967) spectrum of 86Kr then Caesium (1984) hyperfine levels of C133)

Classical	**Quantum**
Real variable $f : X \to \mathbb{R}$	Self-adjoint operator in Hilbert space
Infinitesimal $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$	Compact operator $ds := D^{-1}$
Integral of function $\int f(x)dx$	$⨍ ds^4 =$ coefficient of $\log(\Lambda)$ in $\mathrm{Tr}_\Lambda(ds^4)$

Line Element



Inner automorphisms and internal symmetries

Let us consider the simplest example

$$\mathcal{A} = C^\infty(M, M_n(\mathbb{C})) = C^\infty(M) \otimes M_n(\mathbb{C})$$

Algebra of $n \times n$ matrices of smooth functions on manifold M.

The group $\mathrm{Inn}(\mathcal{A})$ of inner automorphisms is locally isomorphic to the group $\mathcal{G}$ of smooth maps from M to the small gauge group $SU(n)$

$$1 \to \mathrm{Inn}(\mathcal{A}) \to \mathrm{Aut}(\mathcal{A}) \to \mathrm{Out}(\mathcal{A}) \to 1$$

becomes identical to

$$1 \to \mathrm{Map}(M, G) \to \mathcal{G} \to \mathrm{Diff}(M) \to 1.$$

Spectral Action

and Einstein–Yang-Mills

We have shown with A. Chamseddine that the spectral action on this space yields Einstein gravity on M minimally coupled with Yang-Mills theory for the gauge group $SU(n)$. The Yang-Mills gauge potential appears as the inner part of the metric, in the same way as the group of gauge transformations (for the gauge group $SU(n)$) appears as the group of inner diffeomorphisms.

Real Structure J

The restriction to spin manifolds is obtained by requiring a *real structure i.e.* an antilinear unitary operator J acting in $\mathcal{H}$ which plays the same role and has the same algebraic properties as the charge conjugation operator in physics.

In the even case the chirality operator γ plays an important role, both γ and J are decorations of the spectral triple.

The following further relations hold for D, J and γ

$$J^2 = \varepsilon\,,\ \ DJ = \varepsilon' JD, \quad J\,\gamma = \varepsilon''\gamma J, \quad D\gamma = -\gamma D$$

The values of the three signs $\varepsilon, \varepsilon', \varepsilon''$ depend only, in the classical case of spin manifolds, upon the value of the dimension n modulo 8 and are given in the following table :

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

The three roles of J

— In physics J is the charge conjugation operator.

— It is deeply related to Tomita's operator which conjugates the algebra with its commutant. The basic relation always satisfied is Tomita's relation :

$$[a, b^{\mathrm{op}}] = 0\,, \quad \forall a, b \in \mathcal{A}, b^{\mathrm{op}} := J b^* J^{-1}.$$

— KO-homology, one obtains a KO-homology cycle for the algebra $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$ and an intersection form :

$$K(\mathcal{A}) \otimes K(\mathcal{A}) \to \mathbb{Z},\ \mathrm{Index}(D_{e \otimes f})$$

Metric dimension, KO-dimension

In the classical case of spin manifolds there is a relation between the metric (or spectral) dimension given by the rate of growth of the spectrum of D and the integer modulo 8 which appears in the above table. For more general spaces however the two notions of dimension (the dimension modulo 8 is called the KO-dimension because of its origin in K-theory) become independent since there are spaces F of metric dimension 0 but of arbitrary KO-dimension.

Fine Structure

Starting with an ordinary spin geometry M of dimension n and taking the product $M \times F$, one obtains a space whose metric dimension is still n but whose KO-dimension is the sum of n with the KO-dimension of F.

As it turns out the Standard Model with neutrino mixing favors the shift of dimension from the 4 of our familiar space-time picture to $10 = 4 + 6 = 2$ modulo 8.

Finite spaces

In order to learn how to perform the above shift of dimension using a 0-dimensional space F, it is important to classify such spaces. This was done in joint work with A. Chamseddine. We classified there the *finite* spaces F of given KO-dimension. A space F is finite when the algebra $\mathcal{A}_F$ of coordinates on F is finite dimensional. We found among the choices of KO-dimension 6

$$\mathcal{A}_F = M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$$

but we had no uniqueness statement.

Feynman slash of position variables

The role of the position variable q in the higher analogue of $[p, q] = i\hbar$ was the most difficult to uncover.

The answer is to encode the analogue of the position variable q in the same way as the Dirac operator encodes the components of the momenta, just using the Feynman slash.

Motivating examples

Geometry of circle of length 2π :

$$U^*[D,U] = 1$$

Geometry of 2-sphere

$$M_2(\mathbb{C}) \star e,\ e = e^* = e^2$$

Feynman Slash

We let $Y \in \mathcal{A} \otimes C_\kappa$ be of the Feynman slashed form $Y = Y^A \Gamma_A$, and fulfill the equations

$$Y^2 = \kappa, \qquad Y^* = \kappa Y \tag{1}$$

Here $\kappa = \pm 1$ and $C_\kappa \subset M_s(\mathbb{C})$, $s = 2^{n/2}$, is the Clifford algebra on $n+1$ gamma matrices Γ_a, $0 \le a \le n$

$$\Gamma_A \in C_\kappa, \quad \left\{\Gamma^A, \Gamma^B\right\} = 2\kappa\, \delta^{AB},\ (\Gamma^A)^* = \kappa \Gamma^A$$

Higher Heisenberg equation

The one-sided higher analogue of the Heisenberg commutation relations is

$$\frac{1}{n!}\langle Y\,[D,Y]\cdots[D,Y]\rangle = \sqrt{\kappa}\,\gamma \qquad (n \text{ terms } [D,Y]) \qquad (2)$$

where the notation $\langle T\rangle$ means the *normalized* trace of $T = T_{ij}$ with respect to the above matrix algebra $M_s(\mathbb{C})$ ($1/s$ times the sum of the s diagonal terms T_{ii}).

Volume is quantized

For even n, equation (2), together with the hypothesis that the eigenvalues of D grow as in dimension n, imply that the volume, expressed as the leading term in the Weyl asymptotic formula for counting eigenvalues of the operator D, is "quantized" by being equal to the index pairing of the operator D with the K-theory class of $\mathcal{A}$ defined by the projection $e = (1 + \sqrt{\kappa}\, Y)/2$.

Theorem 1 : spheres

Let M be a spin Riemannian manifold of even dimension n and $(\mathcal{A}, \mathcal{H}, D)$ the associated spectral triple. Then a solution of the one-sided equation exists if and only if M breaks as the disjoint sum of spheres of unit volume. On each of these irreducible components the unit volume condition is the only constraint on the Riemannian metric which is otherwise arbitrary for each component.

We recall that given a smooth compact oriented spin manifold M, the associated spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is given by the action in the Hilbert space $\mathcal{H} = L^2(M, S)$ of L^2-spinors of the algebra $\mathcal{A} = C^\infty(M)$ of smooth functions on M, and the Dirac operator D which in local coordinates is of the form

$$D = \gamma^\mu \left(\frac{\partial}{\partial x^\mu} + \omega_\mu \right)$$

where $\gamma^\mu = e_a^\mu \gamma^a$ and ω_μ is the spin-connection

We can assume that $\kappa = 1$ since the other case follows by multiplication by $i = \sqrt{-1}$. Equation (1) shows that a solution Y of the above equations gives a map $Y : M \to S^n$ from the manifold M to the n-sphere. Let us compute the left hand side of (2). The normalized trace of the product of $n+1$ Gamma matrices is the totally antisymmetric tensor

$$\langle \Gamma_A \Gamma_B \cdots \Gamma_L \rangle = i^{n/2} \epsilon_{AB\ldots L}, \quad A, B, \ldots, L \in \{1, \ldots, n+1\}$$

One has

$$[D, Y] = \gamma^\mu \frac{\partial Y^A}{\partial x^\mu} \Gamma_A = \nabla Y^A \Gamma_A$$

where we let ∇f be the Clifford multiplication by the gradient of f. Thus one gets at any $x \in M$ the equality

$$\langle Y\,[D, Y] \cdots [D, Y] \rangle = i^{n/2} \epsilon_{AB\ldots L} Y^A \nabla Y^B \cdots \nabla Y^L$$

Given n operators $T_j \in \mathcal{C}$ in an algebra $\mathcal{C}$ the multiple commutator

$$[T_1, \ldots, T_n] := \sum \epsilon(\sigma) T_{\sigma(1)} \cdots T_{\sigma(n)}$$

(where σ runs through all permutations of $\{1, \ldots, n\}$) is a multilinear totally antisymmetric function of the $T_j \in \mathcal{C}$. In particular, if the $T_i = a_i^j S_j$ are linear combinations of n elements $S_j \in \mathcal{C}$ one gets

$$[T_1, \ldots, T_n] = \mathrm{Det}(a_i^j)[S_1, \ldots, S_n] \tag{3}$$

For fixed A, and $x \in M$ the sum over the other indices

$$\epsilon_{AB\dots L} Y^A \nabla Y^B \cdots \nabla Y^L = (-1)^A Y^A [\nabla Y^1, \nabla Y^2, \dots, \nabla Y^{n+1}]$$

where all other indices are $\neq A$. At $x \in M$ one has $\nabla Y^j = \gamma^\mu \partial_\mu Y^j$ and by (3) the multi-commutator (with ∇Y^A missing) gives

$$[\nabla Y^1, \nabla Y^2, \dots, \nabla Y^{n+1}] = \epsilon^{\mu\nu\dots\lambda} \partial_\mu Y^1 \cdots \partial_\lambda Y^{n+1} [\gamma^1, \dots, \gamma^n]$$

Since $\gamma^\mu = e_a^\mu \gamma_a$ and $i^{n/2}[\gamma_1, \ldots, \gamma_n] = n!\gamma$ one thus gets

$$\langle Y\,[D,Y]\cdots[D,Y]\rangle = n!\gamma \mathrm{Det}(e_a^\alpha)\,\omega$$

where

$$\omega = \epsilon_{AB\ldots L} Y^A \partial_1 Y^B \cdots \partial_n Y^L$$

so that $\omega dx_1 \wedge \cdots \wedge dx_n$ is the pullback $Y^\#(\rho)$ by the map $Y : M \to S^n$ of the rotation invariant volume form ρ on the unit sphere S^n given by

$$\rho = \frac{1}{n!} \epsilon_{AB\ldots L} Y^A dY^B \wedge \cdots \wedge dY^L$$

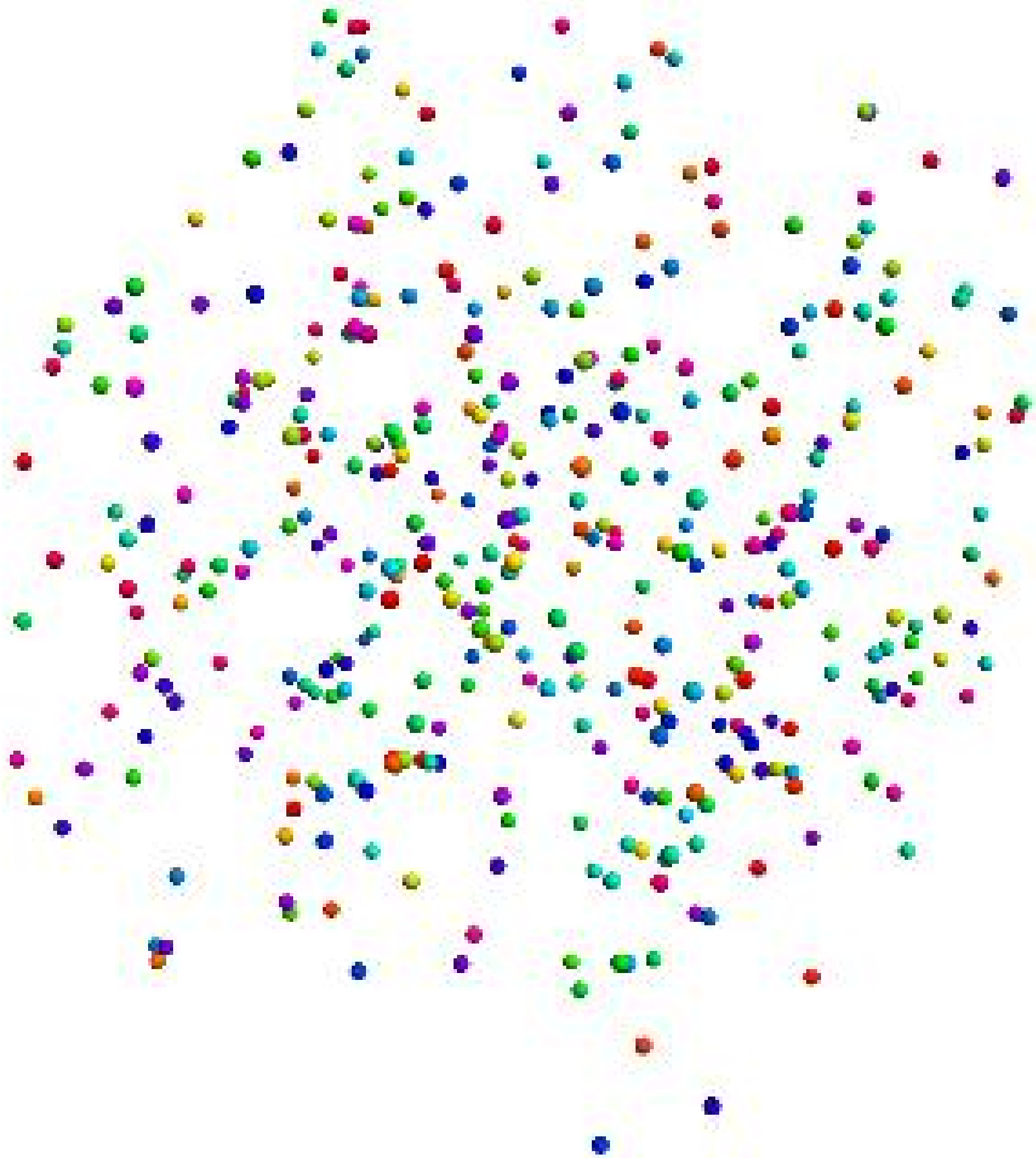
Thus, using the inverse vierbein, the one-sided equation (2) is equivalent to

$$\det\left(e^a_\mu\right) dx_1 \wedge \cdots \wedge dx_n = Y^{\#}(\rho)$$

This equation implies that the Jacobian of the map $Y : M \to S^n$ cannot vanish anywhere, and hence that the map Y is a covering.

Since the sphere S^n is simply connected for $n > 1$, this implies that on each connected component $M_j \subset M$ the restriction of the map Y to M_j is a diffeomorphism. Moreover the equation shows that the volume of each component M_j is the same as the volume $\int_{S^n} \rho$ of the sphere.

Conversely it was shown that, for $n = 2, 4$, each Riemannian metric on S^n whose volume form is the same as for the unit sphere gives a solution to the above equation. In fact the above discussion gives a direct proof of this fact for all (even) n.



Two kinds of quanta

It would seem at this point that only disconnected geometries fit in this framework but this is ignoring an essential piece of structure of the NCG framework, which allows one to refine (2). It is the real structure J, an antilinear isometry in the Hilbert space $\mathcal{H}$ which is the algebraic counterpart of charge conjugation.

Two sided equation

This leads to refine the quantization condition by taking J into account as the two-sided equation

$$\frac{1}{n!}\langle Z\,[D,Z]\cdots[D,Z]\rangle = \gamma \quad Z = 2EJEJ^{-1} - 1, \qquad (4)$$

where E is the spectral projection for $\{1,i\} \subset \mathbb{C}$ of the double slash $Y = Y_+ \oplus Y_- \in C^\infty(M, C_+ \oplus C_-)$. More explicitly $E = \frac{1}{2}(1 + Y_+) \oplus \frac{1}{2}(1 + iY_-)$.

Geometry gives Standard Model !

It turns out that in dimension 4, *i.e.* for 5 gamma :

$$C_+ = M_2(\mathbb{H}), \quad C_- = M_4(\mathbb{C})$$

which give the algebraic constituents of the Standard Model exactly in the form of our previous work ! ! ! !

The two maps $Y_\pm : M \to S^n$

One now gets two maps $Y_\pm : M \to S^n$ while (4) becomes,

$$\det\left(e^a_\mu\right) = \Omega_+ + \Omega_-, \tag{5}$$

with $\Omega_\pm$ the Jacobian of $Y_\pm$ (the pullback of the volume form of the sphere).

Lemma

In the 4-dimensional case one has

$$\left\langle Z\,[D,Z]^4\right\rangle = \frac{1}{2}\left\langle Y\,[D,Y]^4\right\rangle + \frac{1}{2}\left\langle Y'\left[D,Y'\right]^4\right\rangle .$$

In the next theorem the algebraic relations between $Y_\pm$, D, J, $C_\pm$, γ are assumed to hold.

Theorem 2 : $n = 4$

(i) In any operator representation of the two sided equation (4) in which the spectrum of D grows as in dimension 4 the volume (the leading term of the Weyl asymptotic formula) is quantized.

(ii) Let M be a compact oriented spin Riemannian manifold of dimension 4. Then a solution of (5) exists if and only if the volume of M is quantized to belong to the invariant $q_M \subset \mathbb{Z}$.

The invariant $q_M \subset \mathbb{Z}$

$D(M)$ set of pairs of smooth maps $\phi_\pm : M \to S^n$ such that the differential form

$$\phi_+^\#(\alpha) + \phi_-^\#(\alpha) = \omega$$

does not vanish anywhere on M (α is the volume form of sphere S^n).

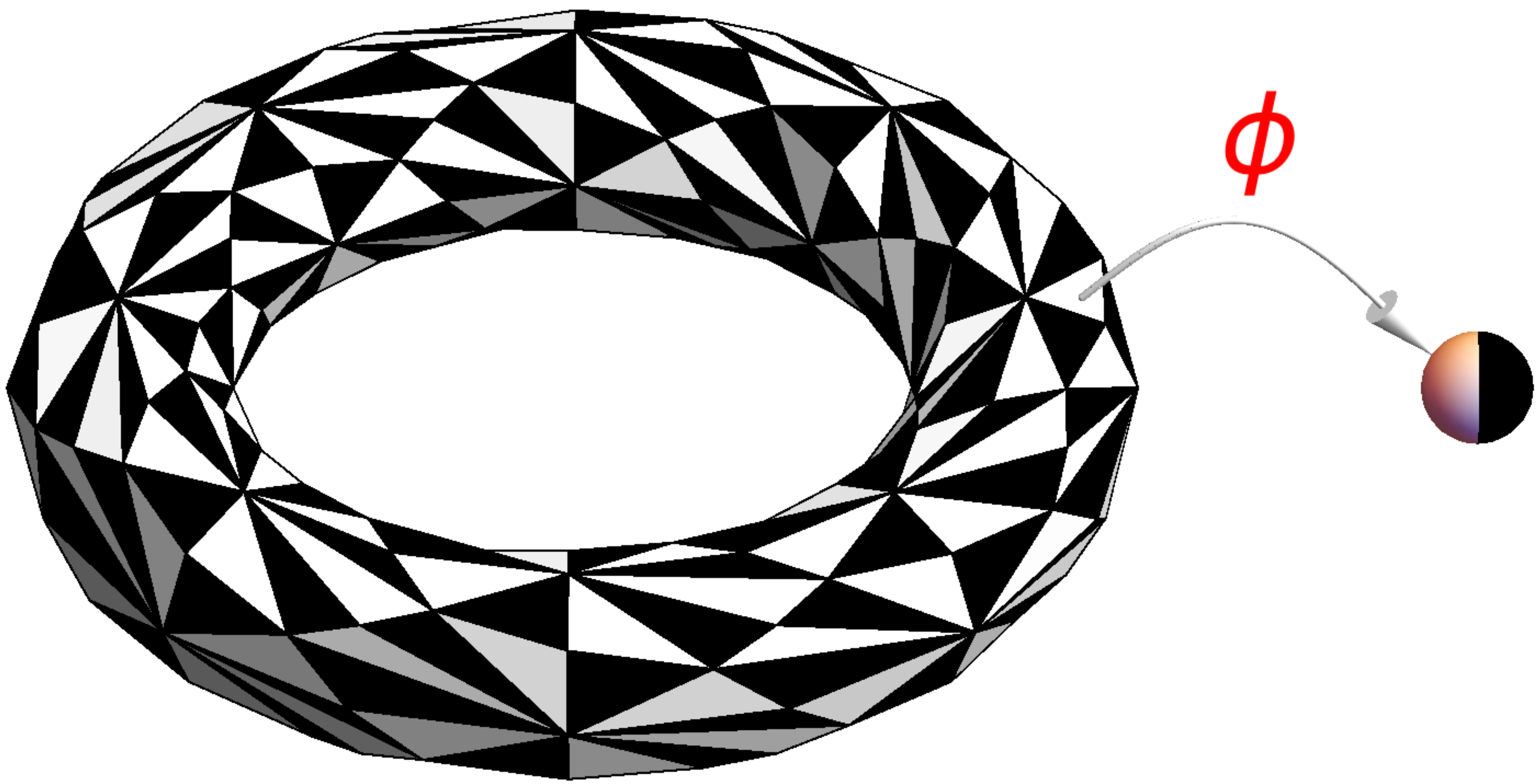
$$q_M := \{\mathrm{deg}(\phi_+) + \mathrm{deg}(\phi_-) \mid (\phi_+, \phi_-) \in D(M)\}$$

where $\mathrm{deg}(\phi)$ is the topological degree of ϕ.

The invariant q_M makes sense in any dimension. For $n = 2, 3$, and any M, it contains all sufficiently large integers. The case $n = 4$ is more difficult and we showed that for any Spin manifold it contains all integers $m > 4$. This uses fine results on existence of ramified covers of the sphere and on immersion theory going back to Smale, Milnor and Poenaru.

Theorem

Let M be a smooth connected oriented compact spin 4-manifold. Then q_M contains all integers $m \geq 5$.


ϕ

Necessary condition

Jean-Claude Sikorav and Bruno Sevennec found the following obstruction which implies for instance that $D(\mathbb{C}P^2) = \emptyset$.

Let M be an oriented compact smooth 4-dimensional manifold, then, with w_2 the second Stiefel-Whitney class of the tangent bundle,

$$D(M) \neq \emptyset \implies w_2^2 = 0$$

One has a cover of M by two open sets on which the tangent bundle is stably trivialized. Thus the product of any two Stiefel-Whitney classes vanishes.

Basic Lemma

Let $\phi : M \to S^4$ be a smooth map such that $\phi^{\#}(\alpha)(x) \geq 0 \ \forall x \in M$ and let $R = \{x \in M \mid \phi^{\#}(\alpha)(x) = 0\}$. Then there exists a map ϕ' such that $\phi^{\#}(\alpha) + \phi'^{\#}(\alpha)$ does not vanish anywhere if and only if there exists an immersion $f : V \to \mathbb{R}^4$ of a neighborhood V of R. Moreover if this condition is fulfilled one can choose ϕ' to be of degree 0.

Spectral Action

The bothering cosmological leading term of the spectral action is now quantized and thus it no longer appears in the variation of the spectral action which now reproduces the Einstein equations coupled with matter. The geometry appears from the joint spectrum of the $Y_{\pm}$ and is a 4-dimensional immersed submanifold in the 8-dimensional product $S^4 \times S^4$. One has the strong Whitney embedding theorem : $M^4 \subset \mathbb{R}^4 \times \mathbb{R}^4 \subset S^4 \times S^4$.

Standard Model	Spectral Action
Higgs Boson	Inner metric$^{(0,1)}$
Gauge bosons	Inner metric$^{(1,0)}$
Fermion masses u, ν	Dirac$^{(0,1)}$ in $\uparrow$
CKM matrix Masses down	Dirac$^{(0,1)}$ in $(\downarrow 3)$
Lepton mixing Masses leptons e	Dirac$^{(0,1)}$ in $(\downarrow 1)$

Standard Model	Spectral Action
Majorana mass matrix	$\text{Dirac}^{(0,1)}$ on $E_R \oplus J_F E_R$
Gauge couplings	Fixed at unification
Higgs scattering parameter	Fixed at unification
Tadpole constant	$-\mu_0^2 \, |\mathbf{H}|^2$

General Case

The components of the connection A which are tensored with the Clifford gamma matrices γ^{μ} are the gauge fields of the Pati–Salam model with the symmetry of $SU(2)_R \times SU(2)_L \times SU(4)$.

The non-vanishing components of the connection which are tensored with the gamma matrix γ_5 are given by

$$(A)_{aI}^{\dot{b}J} \equiv \gamma_5 \Sigma_{aI}^{\dot{b}J}, \quad (A)_{aI}^{b'J'} = \gamma_5 H_{aIbJ}, \quad (A)_{\dot{a}I}^{\dot{b}'J'} \equiv \gamma_5 H_{\dot{a}I\dot{b}J}$$

where $H_{aIbJ} = H_{bJaI}$ and $H_{\dot{a}I\dot{b}J} = H_{\dot{b}J\dot{a}I}$, which is the most general Higgs structure possible.

These correspond to the representations with respect to $SU(2)_R \times SU(2)_L \times SU(4)$:

$$\Sigma_{aI}^{\dot{b}J} = (2_R, 2_L, 1) + (2_R, 2_L, 15)$$
$$H_{aIbJ} = (1_R, 1_L, 6) + (1_R, 3_L, 10)$$
$$H_{\dot{a}I\dot{b}J} = (1_R, 1_L, 6) + (3_R, 1_L, 10)$$

Three models

1) Left-right symmetric Pati–Salam model with fundamental Higgs fields $\Sigma_{aI}^{\dot{b}J}$, H_{aIbJ} and $H_{\dot{a}I\dot{b}J}$. In this model the field H_{aIbJ} should have a zero vev.

2) A Pati–Salam model where the Higgs field H_{aIbJ} that couples to the left sector is set to zero which is desirable because there is no symmetry between the left and right sectors at low energies.

3) If one starts with $(D_F)_{aI}^{\dot{b}J}$ or $(D_F)_{aI}^{b'J'}$ or $(D_F)_{\dot{a}I}^{\dot{b}'J'}$ whose values are given by those that were derived for the Standard Model, then the Higgs fields $\Sigma_{aI}^{\dot{b}J}$, H_{aIbJ} and $H_{\dot{a}I\dot{b}J}$ will become composite and expressible in

terms of more fundamental fields Σ_I^J, $\Delta_{\dot{a}J}$ and $\phi_{\dot{a}}^b$ in the following way :

$$\Sigma_{\dot{a}I}^{bJ} = \left(k^\nu \phi_{\dot{a}}^b + k^e \widetilde{\phi}_{\dot{a}}^b\right) \Sigma_I^J + \left(k^u \phi_{\dot{a}}^b + k^d \widetilde{\phi}_{\dot{a}}^b\right) \left(\delta_I^J - \Sigma_I^J\right)$$

$$H_{\dot{a}I\dot{b}J} = k^{*\nu_R} \Delta_{\dot{a}J} \Delta_{\dot{b}I}.$$

The field $\widetilde{\phi}_{\dot{a}}^b$ is not an independent field and is given by

$$\widetilde{\phi}_{\dot{a}}^b = \sigma_2 \overline{\phi}_{\dot{a}}^b \sigma_2.$$

Depending on the precise particle content we determine the coefficients b_R, b_L, b in

$$16\pi^2 \frac{dg}{dt} = -bg^3$$

that control the RG flow of the Pati–Salam gauge couplings g_R, g_L, g. We run them to look for unification of the coupling $g_R = g_L = g$. The boundary conditions are taken at the intermediate mass scale $\mu = m_R$ to be the usual

$$\frac{1}{g_1^2} = \frac{2}{3}\frac{1}{g^2} + \frac{1}{g_R^2}, \qquad \frac{1}{g_2^2} = \frac{1}{g_L^2}, \qquad \frac{1}{g_3^2} = \frac{1}{g^2},$$

in terms of the Standard Model gauge couplings g_1, g_2, g_3.

At the mass scale m_R the Pati–Salam symmetry is broken to that of the Standard Model, and we take it to be the same scale that is present in the see-saw mechanism. It should thus be of the order $10^{11} - 10^{13}$Gev. We now discuss the three models, in order of complexity.

Pati–Salam with composite Higgs fields

We first consider the case of a finite Dirac operator for which the Standard Model subalgebra $\mathbb{C} \oplus \mathbb{H}_L \oplus M_3(\mathbb{C}) \subset \mathcal{A}_F$ satisfies the first-order condition.

The inner perturbations $\Sigma_{\dot{a}I}^{bJ}$ and $H_{\dot{a}I\dot{b}J}$ are composite and expressible in terms of more fundamental fields Σ_I^J, $\Delta_{\dot{a}J}$ and $\phi_{\dot{a}}^b$

particle	$SU(2)_R$	$SU(2)_L$	$SU(4)$
$\phi_{\dot{a}}^b$	2	2	1
$\Delta_{\dot{a}I}$	2	1	4
Σ_J^I	1	1	15

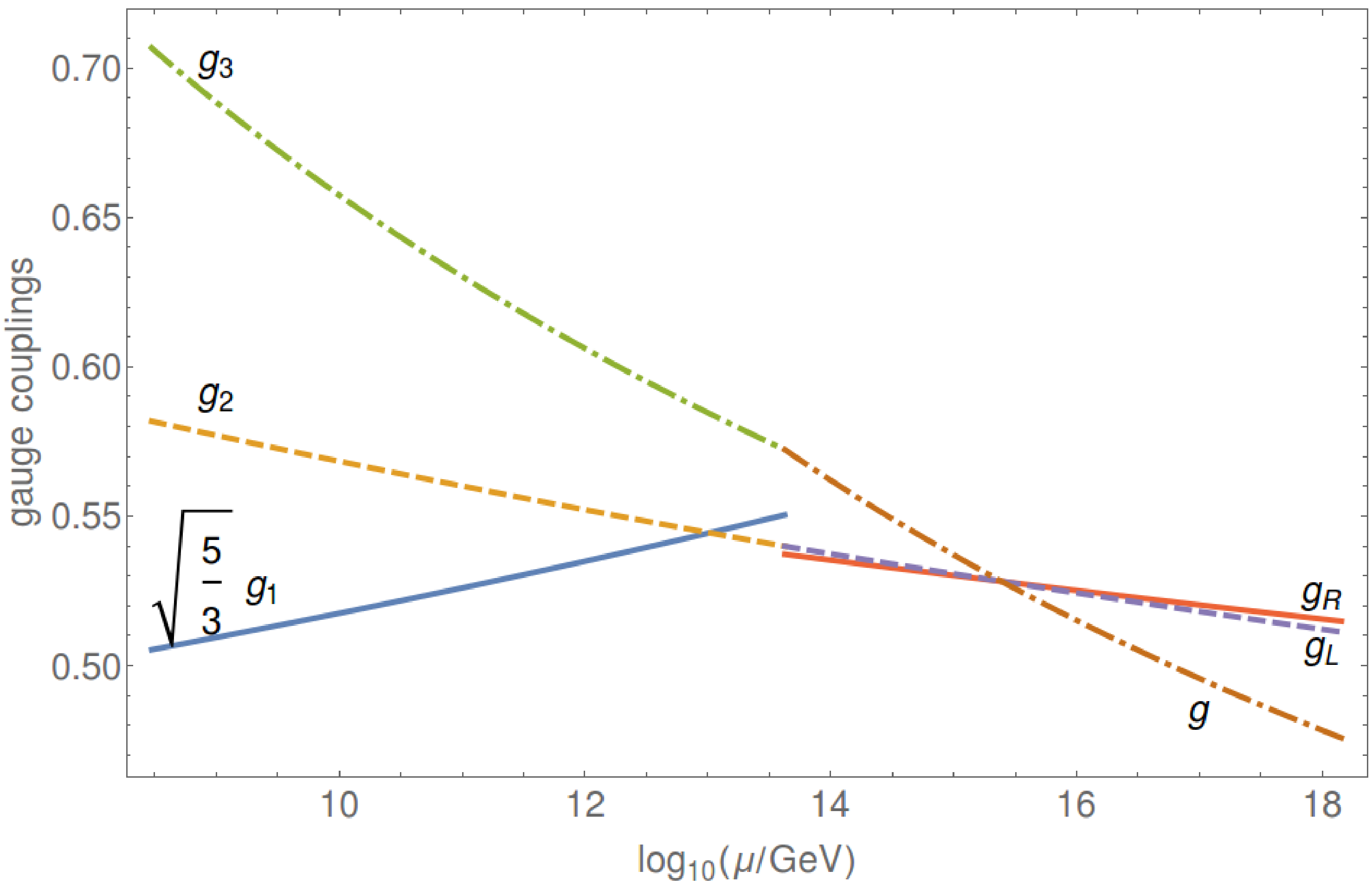
The β-functions for the Pati–Salam couplings g_R, g_L, g with the above particle content are found to be

$$(b_R, b_L, b) = \left(\frac{7}{3}, 3, \frac{31}{3}\right)$$

The solutions of the RG-equations are found to be

$$g_R(\mu)^{-2} = g_R(m_R)^{-2} + \frac{1}{8\pi^2}\frac{7}{3}\log\frac{\mu}{m_R},$$
$$g_L(\mu)^{-2} = g_L(m_R)^{-2} + \frac{1}{8\pi^2}3\log\frac{\mu}{m_R},$$
$$g(\mu)^{-2} = g(m_R)^{-2} + \frac{1}{8\pi^2}\frac{31}{3}\log\frac{\mu}{m_R}$$

We impose the boundary conditions at the mass scale $\mu = m_R$. We find a unification scale $\Lambda \approx 2.5 \times 10^{15}$Gev if we set $m_R = 4.25 \times 10^{13}$Gev



g_3
g_2
$\sqrt{\frac{5}{3}}\, g_1$
g_R
g_L
g
0.70
0.65
0.60
0.55
0.50
gauge couplings
10
12
14
16
18
$\log_{10}(\mu/\mathrm{GeV})$

Pati–Salam with fundamental Higgs fields

Next, we consider the case of a more general finite Dirac operator, not satisfying the first-order condition with respect to the Standard Model subalgebra.

The inner perturbations $\Sigma_{\dot{a}I}^{bJ}$ and $H_{\dot{a}I\dot{b}J}$ are now themselves fundamental Higgs fields and their representations are listed in the following table :

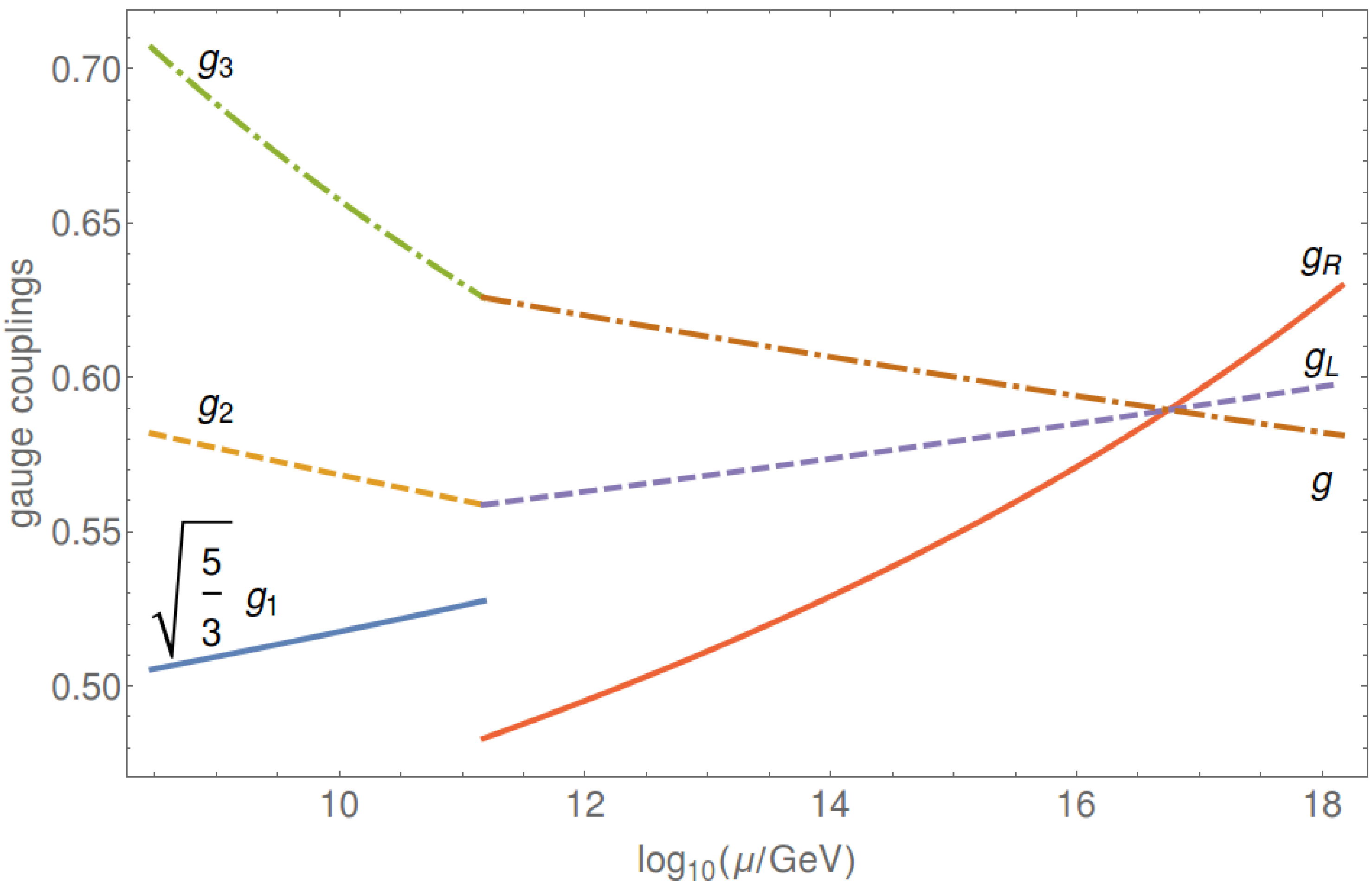
particle	$SU(2)_R$	$SU(2)_L$	$SU(4)$
$\Sigma_{\dot{a}J}^{bJ}$	2	2	$1+15$
$H_{\dot{a}I\dot{b}J}$	3	1	10
	1	1	6

The β-functions are computed to be

$$(b_R, b_L, b) = \left(-\frac{26}{3}, -2, 2\right)$$

Here is the running of coupling constants for the spectral Pati–Salam model with fundamental Higgs fields : g_1, g_2, g_3 for $\mu < m_R$ and g_R, g_L, g for $\mu > m_R$.

The unification scale is $\Lambda \approx 6.3 \times 10^{16}$Gev if we set $m_R = 1.5 \times 10^{11}$Gev.



g_3
g_2
$\sqrt{\frac{5}{3}}\, g_1$
g_R
g_L
g
0.70
0.65
0.60
0.55
0.50
gauge couplings
10
12
14
16
18
$\log_{10}(\mu/\mathrm{GeV})$

Left right symmetric model

As a final possibility we consider the most general case for D_F which gives in addition to the fundamental Higgs fields in the table

particle	$SU(2)_R$	$SU(2)_L$	$SU(4)$
$\Sigma_{\dot{a}J}^{bJ}$	2	2	$1+15$
$H_{\dot{a}I\dot{b}J}$	3	1	10
	1	1	6

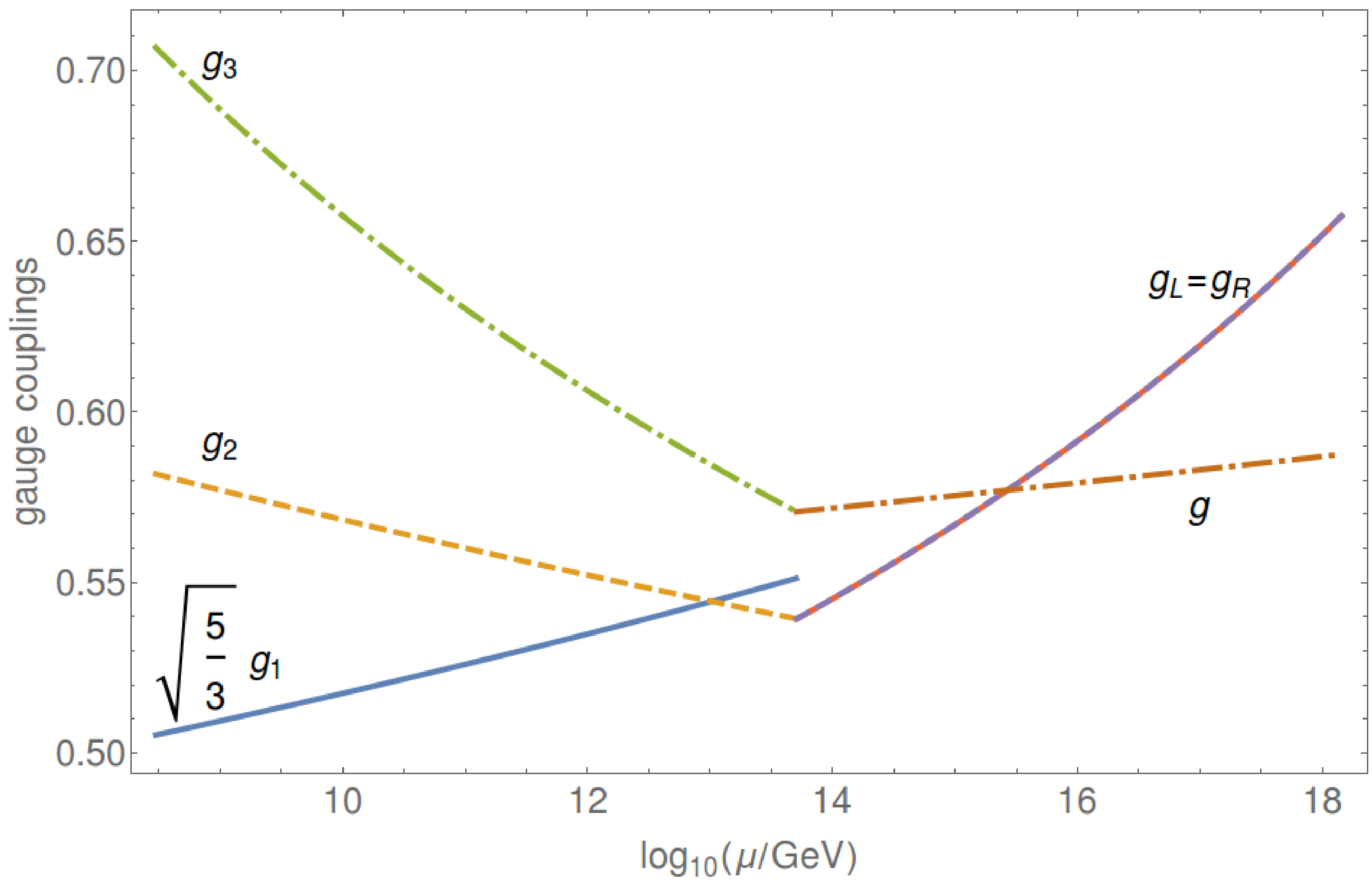
the field H_{aIbJ} in the $(1_R, 3_L, 10) + (1_R, 1_L, 6)$ representation, which gives left-right symmetry.

The β-functions become

$$(b_R, b_L, b) = \left(-\frac{26}{3}, -\frac{26}{3}, -\frac{4}{3}\right)$$

Adopting once more the approximation that we made use of in the previous sections, we run the Pati–Salam gauge couplings from m_R.

We find the unification scale to be $\Lambda \approx 2.7 \times 10^{15}$Gev if we set $m_R = 5.1 \times 10^{13}$Gev.



gauge couplings
0.70
0.65
0.60
0.55
0.50
10
12
14
16
18
log10(μ/GeV)
g3
g2
√(5/3) g1
gL=gR
g

A first shot at QG

Three "variables", in a fixed Hilbert space with fixed representation of $C_{\pm}$, γ, J :

$$(D, Y_{+}, Y_{-})$$

$$\langle Z\,[D, Z] \cdots [D, Z] \rangle = \gamma$$

where $Z = 2EJEJ^{-1} - 1$ and E is the spectral projection for $\{1, i\} \subset \mathbb{C}$ of $Y = Y_{+} \oplus Y_{-}$.

Whitney strong embedding

Let us explain why it is natural from the point of view of differential geometry also, to consider the two sets of Γ-matrices and then take the operators $Y_{\pm}$ as being the correct variables for a first shot at a theory of quantum gravity. The first question which comes in this respect is if, given a compact 4-dimensional manifold M one can find a map $(Y_+, Y_-) : M \to S^4 \times S^4$ which embeds M as a submanifold of $S^4 \times S^4$.

Reconstruction of M

A : It is true that the joint spectrum of the Y_+^a and Y_-^b is of dimension 4 while one has 8 variables.

B : It is it true that the non-commutative integral

$$⨍\gamma \, \langle Y \, [D, Y]^n \rangle$$

remains quantized.

Why joint spectrum of dimension 4

The reason why A holds in the case of classical manifolds is that in that case the joint spectrum of the Y^A and Y'^B is the subset of $S^n \times S^n$ which is the image of the manifold M by the map $x \in M \mapsto (Y(x), Y'(x))$ and thus its dimension is at most n.

The reason why A holds in general is because of the assumed boundedness of the commutators $[D, Y]$ and $[D, Y']$ together with the commutativity $[Y, Y'] = 0$ (order zero condition) and the fact that the spectrum of D grows like in dimension n.

Why is the volume quantized

The reason why B holds in the general case is that all the lower components of the operator theoretic Chern character of the idempotent $e = \frac{1}{2}(1 + Y)$ vanish and this allows one to apply the operator theoretic index formula which in that case gives (up to suitable normalization)

$$2^{-n/2-1} ⨍ \gamma \langle Y \, [D, Y]^n \rangle \, D^{-n} = \mathrm{Index}\,(D_e)$$

This follows from the local index formula of Connes-Moscovici but in fact one does not need the technical hypothesis since, when the lower components of the operator theoretic Chern character all vanish, one can use the non-local index formula in cyclic cohomology and the determination in the 1994 book of the Hochschild class of the index cyclic cocycle.

Vanishing of lower classes

To be more precise one introduces the following trace operation, given an algebra $\mathcal{A}$ over $\mathbb{R}$ (not assumed commutative) and the algebra $M_n(\mathcal{A})$ of matrices of elements of $\mathcal{A}$, one defines

$$\mathrm{tr} : M_n(\mathcal{A}) \otimes M_n(\mathcal{A}) \otimes \cdots \otimes M_n(\mathcal{A}) \to \mathcal{A} \otimes \mathcal{A} \otimes \cdots \otimes \mathcal{A}$$

by the rule, using $M_n(\mathcal{A}) = M_n(\mathbb{R}) \otimes \mathcal{A}$

$$\mathrm{tr}\,((a_0 \otimes \mu_0) \otimes (a_1 \otimes \mu_1) \otimes \cdots \otimes (a_m \otimes \mu_m)) =$$

$$\mathrm{Trace}(\mu_0 \cdots \mu_m) a_0 \otimes a_1 \otimes \cdots \otimes a_m$$

where Trace is the ordinary trace of matrices.

Let us denote by ι_k the operation which inserts a 1 in a tensor at the k-th place. So for instance

$$\iota_0(a_0 \otimes a_1 \otimes \cdots \otimes a_m) = 1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_m$$

One has $\mathrm{tr} \circ \iota_k = \iota_k \circ \mathrm{tr}$ since (taking $k = 0$)

$$\mathrm{tr} \circ \iota_0 \left((a_0 \otimes \mu_0) \otimes (a_1 \otimes \mu_1) \otimes \cdots \otimes (a_m \otimes \mu_m)\right) =$$

$$= \mathrm{tr} \left((1 \otimes 1) \otimes (a_0 \otimes \mu_0) \otimes (a_1 \otimes \mu_1) \otimes \cdots \otimes (a_m \otimes \mu_m)\right)$$

$$= \mathrm{Trace}(1\mu_0 \cdots \mu_m) 1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_m =$$

$$= \iota_0 \left(\mathrm{tr} \left((a_0 \otimes \mu_0) \otimes (a_1 \otimes \mu_1) \otimes \cdots \otimes (a_m \otimes \mu_m)\right)\right)$$

The components of the Chern character of an idempotent $e \in M_s(\mathcal{A})$ are then given up to normalization by

$$\mathrm{Ch}_m(e) := \mathrm{tr}\,((2e-1) \otimes e \otimes e \otimes \cdots \otimes e) \in \mathcal{A} \otimes \mathcal{A} \otimes \ldots \otimes \mathcal{A}$$

with m even and equal to the number of terms e in the right hand side.

Theorem

Let $\mathcal{A}$ be an algebra (over $\mathbb{R}$) and $Y = \sum Y^A \Gamma_A$ with $Y^A \in \mathcal{A}$ and $\Gamma_A \in C_+ \subset M_w(\mathbb{C})$ as above, $n+1$ gamma matrices. Assume that $Y^2 = 1$. Then for any even integer $m < n$ one has $\mathrm{Ch}_m(e) = 0$ where $e = \frac{1}{2}(1+Y)$.

It follows that the component $\mathrm{Ch}_n(e)$ is a Hochschild cycle and that for any cyclic n-cocycle ϕ_n the pairing $< \phi_n, e >$ is the same as $< I(\phi_n), \mathrm{Ch}_n(e) >$ where $I(\phi_n)$ is the Hochschild class of ϕ_n. This applies to the cyclic n-cocycle ϕ_n which is the Chern character ϕ_n in K-homology of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with grading γ where $\mathcal{A}$ is the algebra generated by the components Y^A of Y and Y'^A of Y'.

Local Index

The Hochschild class of ϕ_n is given, up to a normalization factor, by the Hochschild n-cocycle :

$$\tau(a_0, a_1, \dots, a_n) = ⨍ \gamma a_0 [D, a_1] \cdots [D, a_n] D^{-n}, \quad \forall a_j \in \mathcal{A}.$$

Thus one gets that, by the index formula, for any idempotent $e \in M_s(A)$

$$< \tau, \mathrm{Ch}_n(e) > = < \phi_n, e > = \mathrm{Index}\,(D_e) \in \mathbb{Z}$$

Now since D commutes with the two Clifford algebras $C_\pm$, one gets, with $Y = 2e - 1$ as above, the formula

$$< \tau, \mathrm{Ch}_n(e) > = ⨍ \gamma \left\langle Y \, [D, Y]^n \right\rangle D^{-n}$$

The same applies to Y' and we get

Theorem

The quantization equation implies that (up to normalization)

$$⨍ D^{-n} \in \mathbb{N}$$