Étale Difference Algebraic Groups

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Model Theory, Difference/Differential Equations and Applications

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Outline

Motivation: σ-Galois theory of linear differential equations
Joint work with Lucia Di Vizio and Charlotte Hardouin

Difference algebraic geometry

The limit degree and algebraic σ-groups

A decomposition theorem for σ-étale σ-algebraic groups

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Example

The Bessel function $J_\alpha(x)$ solves

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0$$

and satisfies

$$xJ_{\alpha+2}(x) - 2(\alpha + 1)J_{\alpha+1}(x) + xJ_\alpha(x) = 0.$$

The σ-Galois group of Bessel's equation is

$$G = \{g \in \mathrm{SL}_2 \mid \sigma(g) = g\} \leq \mathrm{SL}_2 .$$

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An application of the σ-Galois theory of linear differential equations: σ-independence of special functions

Theorem

Let $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ be two $\mathbb{C}$-linearly independent solutions of $y'' = xy$. Then

$$\mathrm{Ai}(x), \mathrm{Bi}(x), \mathrm{Ai}'(x), \mathrm{Ai}(x+1), \mathrm{Bi}(x+1), \mathrm{Ai}'(x+1), \mathrm{Ai}(x+2), \ldots$$

are algebraically independent over $\mathbb{C}(x)$.

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Difference algebra

Definition

A *difference ring (σ-ring)* is a ring R together with a ring endomorphism $\sigma\colon R \to R$.

Example

$R = \mathbb{C}^{\mathbb{N}}$, $\sigma((a_n)_{n\in\mathbb{N}} = (a_{n+1})_{n\in\mathbb{N}}$

k a σ-field, e.g., $k = \mathbb{C}(\alpha)$ with $\sigma(f(\alpha)) = f(\alpha+1)$. The σ-polynomial ring over k is

$$k\{y\} = k\{y_1,\dots,y_n\} = k[y_1,\dots,y_n,\sigma(y_1),\dots,\sigma(y_n),\sigma^2(y_1),\dots].$$

$F \subset k\{y\}$, R a k-σ-algebra

$$\mathbb{V}_R(F) = \{a \in R^n \mid f(a) = 0 \ \forall\, f \in F\}$$

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Difference varieties

Example

$a_{n+2} = a_{n+1} + a_n \rightsquigarrow \sigma^2(y_1) - \sigma(y_1) - y_1$
$k = \mathbb{C},\ R = \mathbb{C}^{\mathbb{N}} \rightsquigarrow$ Fibonacci-sequence$\in \mathbb{V}_R(\sigma^2(y_1) - \sigma(y_1) - y_1)$

Definition

A functor X of the form $R \rightsquigarrow X(R) = \mathbb{V}_R(F)$ is called a *σ-variety*.

$$\mathbb{I}(X) := \{f \in k\{y\} |\ f(a) = 0\ \forall\ a \in X(R),\ \forall\ R\} \subset k\{y\}$$

$$k\{X\} := k\{y\}/\mathbb{I}(X) \quad \text{coordinate ring of } X$$

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Difference algebraic groups

Definition

A *σ-algebraic group* G is a group object in the category of σ-varieties.

Examples

(Affine) algebraic groups

$$G(R) = \{g \in \mathrm{SL}_2(R) |\ \sigma(g) = g\} \leq \mathrm{SL}_2(R)$$

$$G(R) = \{g \in R^\times |\ g\sigma^2(g)^3 = 1\} \leq \mathbb{G}_m(R)$$

$$G(R) = \{g \in R |\ \sigma^n(g) + \lambda_{n-1}\sigma^{n-1}(g) + \ldots + \lambda_0 y = 0\} \leq \mathbb{G}_a(R)$$

$$G(R) = \{g \in \mathrm{GL}_n(R) |\ g\sigma(g)^\mathsf{T} = \sigma(g)^\mathsf{T} g = I_n\} \leq \mathrm{GL}_n(R)$$

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Difference algebraic groups

Facts:

- The category of σ-varieties is anti–equivalent to the category of finitely σ-generated k-σ-algebras.
- The category of σ-algebraic groups is anti–equivalent to the category of finitely σ-generated k-σ-Hopf algebras.

$$G \leftrightarrow k\{G\}$$

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The limit degree

Theorem

Any σ-algebraic group is isomorphic to a σ-algebraic subgroup of some GL_n.

Fix an embedding $G \hookrightarrow \mathrm{GL}_n$.

$$\mathbb{I}(G) \subset k\{\mathrm{GL}_n\} = k\{X, \tfrac{1}{\det(X)}\}$$

For $i \geq 0$ the ideal

$$\mathbb{I}(G) \cap k[X, 1/\det(X), \ldots, \sigma^i(X), 1/\det(\sigma^i(X))]$$

defines an algebraic subgroup $G[i]$ of GL_n^{i+1} and we have morphisms

$$\pi_i \colon G[i] \to G[i-1], \ (g_0, \ldots, g_i) \mapsto (g_0, \ldots, g_{i-1}).$$

Theorem (Existence of the limit degree)

$\mathrm{ld}(G) = \lim_{i \to \infty} \deg(\pi_i)$ exists and does not depend on the embedding $G \hookrightarrow \mathrm{GL}_n$.

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$0 \leq \alpha_1 < \ldots < \alpha_n$, $1 \leq \beta_1, \ldots, \beta_n$

$$\mathrm{ld}(G) = \beta_n$$

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Algebraic σ-groups in the sense of Kowalski and Pillay

Definition (Kowalski, Pillay)

An algebraic σ-group is an algebraic group G together with a morphism of algebraic groups $\sigma\colon G \to {}^{\sigma}G$.

Theorem

The category of affine algebraic σ-groups is equivalent to the category of σ-algebraic groups of limit degree one (and σ-dimension zero).

Idea of proof: $\mathrm{ld}(G) = 1 \Leftrightarrow k\{G\}$ is finitely generated as a k-algebra.

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A σ-algebraic group G is *σ-étale* if $k\{G\}$ is a union of étale k-algebras $\Leftrightarrow$ every element of $k\{G\}$ satisfies a separable polynomial over k.

Examples

$G(R) = \{g \in R^\times \mid g^n = 1\} \leq \mathbb{G}_m(R)$ iff $\mathrm{char}(k) \nmid n$

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$$1 \to G^o \to G \to G/G^o \to 1$$

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The category of σ-étale σ-algebraic groups

In algebraic geometry:

The category of étale algebraic groups is equivalent to the category of finite groups equipped with a continuous action of the absolute Galois group.

In σ-algebraic geometry:

The category of σ-étale σ-algebraic groups is equivalent to the category of pro-finite σ-groups of finite σ-type equipped with a σ-continuous action of the absolute Galois group.

A pro-finite group group $\mathcal{G}$ with a continuous endomorphism $\sigma \colon \mathcal{G} \to \mathcal{G}$ is of finite σ-type if there exists an open normal subgroup $\mathcal{N}$ of $\mathcal{G}$ such that $\cap_{i\geq 0}\sigma^{-i}(\mathcal{N}) = 1$.

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A decomposition theorem for σ-étale σ-algebraic groups

Theorem

Let G be a σ-étale σ-algebraic group. Then there exists a subnormal series

$$G \supset G_1 \supset G_2 \supset \cdots \supset G_n \supset 1$$

such that $G_1 = G^{\sigma o}$, G_i/G_{i+1} is benign for $i = 1, \ldots, n-1$ and G_n is σ-infinitesimal.

The σ-topology

For a σ-ring R we have an induced map $\sigma\colon \operatorname{Spec}(R) \to \operatorname{Spec}(R)$

Definition

A subset of $\operatorname{Spec}(R)$ is *σ-closed* if it is closed and stable under σ.

$$\mathfrak{a} \mapsto \mathcal{V}(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec}(R) |\ \mathfrak{a} \subset \mathfrak{p}\}$$

is a bijection between the radical σ-ideals of R and the σ-closed subsets of $\operatorname{Spec}(R)$.

Definition

$\operatorname{Spec}(R)$ is *σ-connected* if it is connected with respect to the σ-topology.

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σ-connectedness

In algebraic geometry:

Spec(R) is connected if and only if R has no non-trivial idempotent elements.

In σ-algebraic geometry:

Spec(R) is σ-connected if and only if R has no non-trivial periodic ($\sigma^n(e) = e$) idempotent elements.

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The σ-identity component

Definition

A σ-algebraic group G is *σ-connected* if $\mathrm{Spec}(k\{G\})$ is σ-connected.

Examples

connected σ-algebraic groups, algebraic groups

The σ-connected σ-algebraic subgroup

$$G^{\sigma o}$$

of G corresponding to the σ-connected component of $\mathrm{Spec}(k\{G\})$ that contains the identity, is called the *σ-identity component* of G.

Theorem

$\mathrm{Spec}(k\{G\})$ has only finitely many σ-connected components.

$\Rightarrow \dim_k(k\{G/G^{\sigma o}\}) < \infty$

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σ-infinitesimal σ-algebraic groups

In algebraic geometry:

An algebraic group G is *infinitesimal* if $G(R) = 1$ for every reduced k-algebra R.

In σ-algebraic geometry:

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G σ-infinitesimal $\Leftrightarrow$ the reflexive closure of the zero ideal of $k\{G\}$ defines the trivial group.

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A decomposition theorem for σ-étale σ-algebraic groups

Definition

A σ-algebraic group is *benign* if it is isomorphic to an étale algebraic group (interpreted as a σ-algebraic group).

Theorem

Let G be a σ-étale σ-algebraic group. Then there exists a subnormal series

$$G \supset G_1 \supset G_2 \supset \cdots \supset G_n \supset 1$$

such that $G_1 = G^{\sigma o}$, G_i/G_{i+1} is benign for $i = 1, \ldots, n-1$ and G_n is σ-infinitesimal.

Idea of proof: Induction on $\mathrm{ld}(G)$
Main step: G σ-connected, $\nexists$ $N \trianglelefteq G$ with $1 < \mathrm{ld}(N) < \mathrm{ld}(G)$
$\Rightarrow \exists$ $H \leq G$ σ-infinitesimal : G/H is benign.

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Example

$G(R) = \{g \in R^\times \mid g^4 = 1,\ \sigma(g)^2 = 1\} \leq \mathbb{G}_m(R)$

$2 \nmid \mathrm{char}(k) \Rightarrow G$ σ-étale
$G = G^{\sigma o}$
$N(R) = \{g \in R^\times \mid g^4 = 1,\ \sigma(g) = 1\} \Rightarrow N \trianglelefteq G$ σ-infinitesimal

$$G = G^{\sigma o} \supset N \supset 1$$

$H(R) = \{g \in R^\times \mid g^2 = 1\} \leq \mathbb{G}_m(R)$

$$G \to H,\ g \mapsto \sigma(g)$$

is surjective with kernel $N \Rightarrow G/N \simeq H$ is benign.

Thank you!

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