Toward quantum Picard-Vessiot Theory

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1 Introduction

Hypergeometric series

$$F\left[\begin{matrix} a & b \\ c & \end{matrix}; z\right] = 1 + \frac{a\, b}{c\, 1} z + \frac{a(a+1)\, b(b+1)}{c(c+1)\, 1\, 2} z^2 + \cdots$$

$$z(1-z)\frac{d^2F}{dz^2} + (c-(a+b+1)z)\frac{dF}{dz} - ab\,F = 0$$

$q \in \mathbb{C}$, $q \neq 0, 1$

q-hypergeometric series

$$F\left[\begin{matrix} \alpha, & \beta \\ \gamma & \end{matrix}, q; z\right] =$$

linear difference equation $/ \mathbb{C}(z)$

$q \to 1 \quad F\left[\begin{matrix}\alpha\ \beta \\ \gamma\end{matrix}, q; z\right] = F\left[\begin{matrix}a\ b \\ c\end{matrix}, z\right]$

q-hypergeometric ← hypergeometric Quantization

Galois group of q-hypergeometric function is a linear algebraic group
Galois group of hypergeometric function is a linear algebraic group

Galois group is not quantized. Picard-Vessiot theory.

$\exists$? Quantized Picard-Vessiot theory? Question

Galois group is a quantum group

Differential P.-V. theory Hopf algebra $C[G_a]$

Difference P.-V. theory Hopf algebra $C[G_m]$

$\Downarrow$

Hopf Galois theory Replace $C[G_a]$, $C[G_m]$ by a general Hopf algebra H.

(Sweedler, Takeuchi, Masuoka, Amano ...)

H might be highly non-commutative.

$\uparrow$ operators

Mainly they assume H operates commutative algebras $\Longrightarrow$ So far Galois group in Hopf P.-V. theory is a linear alg. gr.

Our result

Linear psi-equations q-skew iterative σ-differential

$\sigma, \theta^{(1)} : R \to R$ linear operators on an algebra $R/\mathbb{C}$

(1) σ is an automorphism.

(2) $\theta^{(1)}(xy) = \theta^{(1)}(x)\, y + \sigma(x)\, \theta^{(1)}(y)$ for $\forall x, y \in R$

(3) $\theta^{(1)}\sigma = q\sigma\theta^{(1)}$, $q \in \mathbb{C}$.

Quantization happens for linear psi-equations with constant coefficients

First small step.

Relation with the preceding theories :

(1) $\sigma = \mathrm{Id}$, $q=1$ Differential Galois theory P.-V. linear alg. group

(2) $\theta^{(1)} = 0$, Difference Galois theory P.-V. linear alg. group

(3) general, $R \supset \mathbb{C}(t)$ σ = q-difference isomorphism P.-V. linear alg. group.

R commutative

$$\theta^{(1)} = \frac{\sigma - \mathrm{Id}}{(q-1)t}$$

Hardouin 2010 Masuoka Yanagawa 2013

(4) $/\mathbb{C}$ Quantized P.-V. theory Quantum groups

So far they considered commutative rings and they have not encountered quantum groups.

To quantize Galois theory, we have to quantize

the space = the ring of functions.

What is it good for?

Look for non-commutative functional equations

Noumi 1991

Quantization of hypergeometric series

Gelfand's theory of general hypergeometric functions

They live on the Grassmannians $Gr(m,n)$

$\Downarrow$ Noumi starts by quantizing Grass $Gr(m,n)$

general q-hypergeometric functions

linear functional equations over a non-commutative ring.

He observed only their shadows over commutative rings!

Galois theoretic understanding of Noumi's non-commutative q-hypergeometric functions

$$\Uparrow$$

Quantized Picard-Vessiot theory

Motivation

§2 qsi modules and qsi algebras

$q \in \mathbb{C}$, $q \neq 0$, $q^n \neq 1$ $n = 1, 2, 3, \cdots$

(1) $\mathbb{C}(t)/\mathbb{C}$, d/dt, P.-V. extension, Galois group G_a $dt/dt = 1$, $d^2t/dt = 0$

(2) $\mathbb{C}(t)/\mathbb{C}$, $\sigma : \mathbb{C}(t) \to \mathbb{C}(t)$, P.V. extension, G_m $\sigma(t) = qt$
$t \mapsto qt$

(3) Difference differential $\mathbb{C}(t)/\mathbb{C}$ $\begin{cases} \sigma(t) = qt \\ \theta^{(1)}(t) = 1 \end{cases}$

$\sigma : \mathbb{C}(t) \to \mathbb{C}(t)$, $t \mapsto qt$,

$$\theta^{(1)}(f(t)) = \frac{f(qt) - f(t)}{(q-1)t} : \mathbb{C}(t) \to \mathbb{C}(t) \quad \mathbb{C}\text{-linear}$$

$(\mathbb{C}(t), \sigma, \theta^{(*)})/\mathbb{C}$ is not a P.-V. ext.

Galois group of the normalization is a Hopf algebra

h_q that is neither commutative nor co-commutative.

q-skew iterative σ-differential (difference)

qsi = q-skew iterative σ-differential

Example of a qsi module, $\mathbb{C}\langle\sigma, \sigma^{-1}, \theta^{(1)}\rangle$-module $q\sigma\theta^{(1)} = \theta^{(1)}\sigma$.

Imagine $m_1 = t$, $m_2 = 1$ in $\mathbb{C}[t]$

$M = \mathbb{C}t \oplus \mathbb{C}\cdot 1 = \mathbb{C}m_1 \oplus \mathbb{C}m_2$

$$\sigma\left(\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}\right) = \begin{bmatrix} qm_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}, \qquad \sigma\left(\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}\right) = A\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$$

$$\theta^{(1)}\left(\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}\right) = \begin{bmatrix} m_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}, \qquad \theta^{(1)}\left(\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}\right) = B\begin{bmatrix} m_1 \\ m_2 \end{bmatrix},$$

$$A = \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \qquad AB = qBA$$

We consider qsi linear equation

$$\sigma(y) = Ay, \quad \theta^{(1)}(y) = By \qquad (*)$$

To solve qsi equation $(*)$ = to find qsi algebra R and $\mathbb{C}[\sigma, \theta^{(1)}]$-module morphism

$$f : M \longrightarrow R.$$

R is a $\mathbb{C}[\sigma, \theta^{(1)}]$-module such that

$$\theta^{(1)}(ab) = \theta^{(1)}(a)b + \sigma(a)\theta^{(1)}b. \quad \text{for } \forall a, b \in R$$

We set $\theta^{(0)} = Id_{\mathbb{C}(t)}$, $\theta^{(i)} = \frac{1}{[i]_q!}(\theta^{(1)})^i : \mathbb{C}(t) \to \mathbb{C}(t)$ $\quad i = 1, 2, 3, \cdots$

where $[i]_q = 1 + q + \cdots + q^{i-1}$ for $i = 1, 2, \cdots$

$[i]_q! = [i]_q [i-1]_q \cdots [1]_q$.

$q^i \sigma \theta^{(i)} = \theta^{(i)} \sigma$, $\quad \theta^{(m)} \cdot \theta^{(n)} = \binom{m+n}{m}_q \theta^{(m+n)}$

$$\theta^{(n)}(ab) = \sum_{i+j=n} \sigma^i \theta^{(i)}(a) \theta^{(j)}(b).$$

$H_q = \mathbb{C}\langle \sigma, \sigma^{-1}, \theta^{(1)} \rangle$ is a Hopf algebra

$\Delta : H_q \to H_q \otimes_{\mathbb{C}} H_q$

$\Delta(\sigma) = \sigma \otimes \sigma$, $\Delta(\sigma^{-1}) = \sigma^{-1} \otimes \sigma^{-1}$, $\Delta(\theta^{(1)}) = \theta^{(1)} \otimes 1 + \sigma \otimes \theta^{(1)}$.

Look for solutions $y=\begin{bmatrix}y_{11}\\ y_{21}\end{bmatrix}, \begin{bmatrix}y_{12}\\ y_{22}\end{bmatrix} \in R^2$ for (*)

(*) $\quad \sigma(y)=Ay, \quad \theta^{(1)}(y)=By$

Lemma If R is a commutative qsi algebra. Then $\det[y_{ij}]=0$.

Proof

$$\sigma(y_{11})=qy_{11} \qquad \sigma(y_{12})=qy_{12}, \ \theta^{(1)}(y_{11})=y_{21}, \ \theta^{(1)}(y_{12})=y_{22},$$

$$\sigma(y_{21})=y_{21} \qquad \sigma(y_{22})=y_{22}, \ \theta^{(1)}(y_{21})=0, \ \theta^{(1)}(y_{22})=0.$$

$$\theta^{(1)}(y_{11}y_{12})=\theta^{(1)}(y_{11})y_{12}+\sigma(y_{11})\theta^{(1)}(y_{12})=y_{21}y_{12}+qy_{11}y_{22}$$

$$\parallel$$

$$\theta^{(1)}(y_{12}y_{11})=\theta^{(1)}(y_{12})y_{11}+\sigma(y_{12})\theta^{(1)}(y_{11})=y_{22}y_{11}+qy_{12}y_{21}$$

$$(q-1)(y_{11}y_{22}-y_{12}y_{21})=0 \quad \text{So } \det=0$$

We cannot find two linearly independent solutions of (*)
in a commutative qsi algebra R.

3 Fundamental system of solutions of (*)

$Y \in M_2(R)$, R is a qsi algebra

$\sigma(Y) = \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix} Y$, $\quad \partial^{(1)}(Y) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} Y \qquad (*)$

$Y = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$

2-7 = 3-1

$R \otimes_{\mathbb{C}} M$ is a $R[\sigma, \sigma^{-1}; \theta^{(m)}]$-module

<u>Definition</u> M is trivialized over R if $\exists\, c_1, c_2 \in R \otimes_{\mathbb{C}} M$ such that

$R \otimes_{\mathbb{C}} M \cong Rc_1 \oplus Rc_2$ and $\sigma(c_i) = c_i$, $\theta^{(m)}(c_i) = 0$ for $i = 1, 2$.

<u>Lemma</u> Equivalent conditions

(1) M is trivialized over R.

(2) $\exists Z \in M_2(R)$. Z is invertible $Z \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \in (R \otimes_{\mathbb{C}} M)^2$, $\sigma(Z) = ZA^{-1}$

$\theta^{(m)}(Z) = -ZAB$.

(3) $\exists Y \in M_2(R)$, Y is invertible, $\sigma(Y) = AY$, $\theta^{(m)}(Y) = BY$

In (2) $\Leftrightarrow$ (3) $Y^{-1} = Z$, $Z = Y^{-1}$

<u>Definition</u> $Y \in M_2(R)$ is a fundamental system of solutions if

Y satisfies (3).

4 P.-V. extension of (*)

Let Q, t be variables$/\mathbb{C}$ commutation relation $q t Q = Q t$.

$\mathbb{C}\langle t, Q, Q^{-1}\rangle \subset \mathbb{C}(Q)[t] \subset \mathbb{C}(Q)[[t]][t^{-1}]$ division algebra

$q t Q = Q t$, $t Q^{-1} = q Q^{-1} t$, $\sigma(t) = q t$, $\sigma(Q) = q Q$, $\sigma(Q^{-1}) = q^{-1} Q^{-1}$,

$\ldots = Q^{-1}(Q^{-1}) = 0$ $\quad 3-3 = 4-1$

$\mathbb{C}\langle t, Q, Q^{-1}\rangle$ is a psi algebra. $Y := \begin{bmatrix} Q & t \\ 0 & 1 \end{bmatrix} \in M_2(R)$

$$\sigma(Y) = \begin{bmatrix} qQ & qt \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix} Y, \quad \theta^{(1)}(Y) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} Y.$$

$$Y^{-1} = \begin{bmatrix} Q^{-1} & -Q^{-1}t \\ 0 & 1 \end{bmatrix}$$ invertible

$$\mathbb{C}\langle Y, Y^{-1}\rangle = \mathbb{C}\langle Q, Q^{-1}, t\rangle$$

Extension generated by a fundamental system of solution Y

$R = \mathbb{C}\langle t, Q, Q^{-1}\rangle / \mathbb{C}$ is the P.-V. extension of (*)

(1) R is qui simple algebra.

(2) R trivializes M.

(3) $C_R = \mathbb{C}$ No increase of constants.

(4) Uniqueness.

(5) Tensor category. Galois group.

5 Galois group

$\mathbb{C}\langle e, e^{-1}, f\rangle_q = \mathcal{H}_q$ Commutation relations

$ef = qfe$

Hopf algebra $\Delta : \mathcal{H}_q \to \mathcal{H}_q \otimes_{\mathbb{C}} \mathcal{H}_q$

$\Delta(e) = e \otimes e$, $\Delta(e^{-1}) = e^{-1} \otimes e^{-1}$; $\Delta(f) = f \otimes 1 + e \otimes f$

$S : \mathcal{H}_q \to \mathcal{H}_q$ $S(e^{\pm 1}) = e^{\mp 1}$, $S(f) = -e^{-1}f$

$\mathcal{H}_q$ right coacts on R

$R \longrightarrow R \otimes_{\mathbb{C}} \mathcal{H}_q$ $\quad$ σ algebra hom.

$\begin{bmatrix} Q & t \\ 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} Q & t \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} e & f \\ 0 & 1 \end{bmatrix}$ in $Q \mapsto Q \otimes e$, $t \mapsto Q \otimes f + t \otimes 1$

$\mathcal{C}$ = Category of gsi modules of finite dim$/\mathbb{C}$

($\mathbb{C}[\sigma, \sigma^{-1}, \theta^{(1)}]$-modules)

$M_1, M_2 \in \mathrm{ob}\,\mathcal{C}$ $\quad M_1 \otimes_{\mathbb{C}} M_2,\ M_1^{\vee} \in \mathrm{ob}\,\mathcal{C}$ rigide tensor category

$M = \mathbb{C}m_1 \oplus \mathbb{C}m_2$ $\quad \{\{M\}\} \subset \mathcal{C}$ rigide tensor category generated by M.

<u>Theorem</u> Equivalence of rigide tensor categories

$\{\{M\}\} \simeq$ Tensor category of right h_q-comodules of finite dim.

Correspondence. $N \in ob(C\{\{M\}\})$

$R \otimes_{\mathbb{C}} N$ is a qsi-module (left module)

$R \otimes_{\mathbb{C}} N \longrightarrow R \otimes_{\mathbb{C}} N \otimes \mathfrak{h}_q$ right coaction of $\mathfrak{h}_q$ on R

qsi constants of $R \otimes_{\mathbb{C}} N = \hat{N}$ has a right $\mathfrak{h}_q$-comodule structure.

$N \longmapsto \hat{N}$

6 qsi simplicity of $R = \mathbb{C}\langle Q, Q^{-1}, t\rangle$

<u>Proposition</u> R is qsi-simple. No non-trivial bilateral qsi invariant ideal.

Proof $I \subset R$ a bilateral ideal qsi invariant.

$0 \neq f \in I \subset \mathbb{C}[Q, Q^{-1}][t]$

$$f = \sum_{n=0}^{N} a_n(Q, Q^{-1}) t^n$$

$\theta^{(1)}(Q) = 0, \quad \theta^{(1)}(t^n) = [n]_q t^{n-1}$

Applying $\theta^{(1)}$, we may assume

$$0 \neq f = a(Q, Q^{-1}) \in \mathbb{C}[Q, Q^{-1}]$$

Multiplying a power of Q, $0 \neq f \in \mathbb{C}[Q]$

May assume

$$f = a_0 + a_1 Q + \cdots + a_n Q^n \in I \qquad a_i \in \mathbb{C} \text{ for } 1 \le i \le n \qquad a_0 \neq 0$$

Applying $\sigma, \sigma^2, \ldots$

$$\sum_i a_i \zeta^i Q^i = I_1 \in I$$

$$\sum_i a_i \zeta^{2i} Q^i = I_2 \in I$$

$$\vdots$$

Invertible

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \zeta & \cdots & \zeta^n \\ \vdots & & & \\ 1 & \zeta^n & \cdots & \zeta^{n^2} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 Q \\ \vdots \\ a_n Q^n \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_{n+1} \end{bmatrix} \in I^n$$

$\therefore\ a_0 \in I \qquad \therefore\ I = (1).$

1-2

7 Uniqueness

Proposition Let Y_1 be another fundamental system of solutions of (*). We set $S = \mathbb{C}\langle Y_1, Y_1^{-1}\rangle$. If $C_S = \mathbb{C}$, then $R \simeq S$.

Proof $Y_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(S)$. It follows from (*) $c, d \in C_S = \mathbb{C}$.

We may assume $c=0$, $d=1$

$$Y_1 = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}.$$

Sufficient to show $ab = qba$.

In fact, if so, we have a morphism $\underset{\mathbb{C}\langle \Delta, \Delta^{-1}, t\rangle}{\overset{\parallel}{R}} \twoheadrightarrow S$, $\Delta \mapsto a$, $t \mapsto b$

Since R is $\sigma\partial$ simple, $R \simeq S$.

Have to show $ab = q\,ba$. This is not true. We must kill a cocycle.

$$f := q a^{-1} b - b a^{-1} \quad \text{Then } \sigma(f) = f, \quad \partial^{(1)}(f) = 0 \quad f \in \mathbb{C}$$

$$g := \frac{f}{1-q} \in \mathbb{C} \quad \text{and} \quad b' := b + ga$$

Then $Y_2 := \begin{bmatrix} a & b' \\ 0 & 1 \end{bmatrix}$ $\quad ab' = q b' a$.

$$\sigma(Y_2) = AY_2, \quad \partial^{(1)}(Y_2) = BY_2$$

$$S = \mathbb{C}\langle Y_2, Y_2^{-1} \rangle.$$

8 General formulation by Masuoka

H = a Hopf algebra $/\mathbb{C}$.

$H^* = \mathrm{Hom}_{\mathbb{C}}(H, \mathbb{C})$ is a Hopf algebra. The dual of H.

$\exists$ a smaller dual $H^\circ \subset H^*$. Namely, let $I \subset H$ be a bilateral ideal of H such that dim. of the $\mathbb{C}$-vector space H/I is finite.

H/I an algebra $H \to H/I$ algebra morphism.

Dual morphism

$(H/I)^* \longrightarrow H^*$ co-algebra morphism.

$$H^\circ := \varinjlim_{I} (H/I)^* \subset H^* \text{ is a co-algebra}$$

$\dim_k H/I < \infty$

One can show H° is a Hopf algebra

Have to check $\quad I, J \subset H$ two bilateral ideas

$$\begin{tikzcd} H \arrow[r] \arrow[d] \arrow[dr] & H \otimes_k H \arrow[d] \\ \exists\, H/K \arrow[r] & H/I \otimes H/J \end{tikzcd}$$

$\exists K \subset H$ bilateral ideal

Equivalently, we consider a representation

$$\pi : H \to \mathrm{End}(V) = M_n(\mathbb{C}), \quad n = \dim V < \infty$$

π is a morphism of algebras.

So $(\mathrm{End}\, V)^* = M_n(\mathbb{C})^* \to H^\circ$ co-algebra morphism

If we choose a basis of V, $e_1, e_2, \cdots e_n$ $\quad \mathrm{End}\, V \simeq M_n(\mathbb{C})$

canonical basis $e_{ij} \in M_n(\mathbb{C})$ $\quad e_{ij} = \sum_{k=1}^{n} e_{ik} e_{kj}$, $1 \leq i, j \leq n$

$e_{ij}^* : M_n(\mathbb{C}) \to \mathbb{C}$ dual basis

$$M_n(\mathbb{C})^* \longrightarrow H^\circ \quad \text{co-algebra morphism}$$

We may regard $e_{ij}^* \in H^\circ$. $\quad \Delta(e_{ij}^*) = \sum_{k=1}^{n} e_{ik}^* \otimes e_{kj}^*$ $\quad 1 \leq i, j \leq n$

$c_{ij}^* \in H^\circ \subset H^*$

$$\Delta(c_{ij}^*) = \sum_{k=1}^{n} c_{ik}^* \otimes c_{kj}^*$$

$Y_\pi^* = (c_{ij}^*) \in M_n(H^\circ)$ Co-representation matrix

Masuoka

Fundamental system = Co-representation matrix Y_π^*

left H-module algebra structure on $\mathbb{C}\langle Y_\pi^*, Y_\pi^{*-1} \cdots \rangle$?

H^0 = Subalgebra generated by the c^π_{ij}'s for $\forall$ representation π

Tannaka Category of left H-modules $\dim < \infty$

$\simeq$ Category of right H^0-comodules $\dim < \infty$

Equivalence of rigide tensor categories

Correspondence Let N be a right H^0-comodule of $\dim_{\mathbb{C}} N < \infty$

$N \to N \otimes_{\mathbb{C}} H^0$

$H \otimes N \to \underbrace{H \otimes N \otimes H^0} \to N$

Conversely M = a left H-module of finite dim.

Choose a basis $c_1, c_2, \cdots, c_n$ of M. It determines a basis

c_{ij} of End M. Right co-module structure is given by

$$M \to M \otimes H^{\circ}.$$
$$c_i \longmapsto \sum_{j=1}^{n} c_j \otimes c_{ji}^{*}$$

Let M be a left H-module, $\dim_{\mathbb{C}} M < \infty$
choose a basis of M It determines the matrix

$$(c_{ij}^*) \in M_n(H^0)$$

of corepresentation matrix

$H_\pi :=$ The smallest Hopf algebra containing the c_{ij}^*'s $\subset H^0$

Theorem 1 The rigide tensor category $\{\{M\}\}$ generated by M is equivalent to the rigide tensor category of right H_π-co-modules.

Galois group of H-module $M/\mathbb{C}$ is the Hopf algebra H_π.

H_π is a right H°-comodule algebra $\Rightarrow$ $\exists$ left H-module algebra structure of H_π.
denoted by $\overline{H_\pi}$

Theorem (Masuoka) (1) H-module algebra $\overline{H}_\pi$ is simple.

(2) H-constants of $\overline{H}_\pi = \mathbb{C}$.

(3) M is trivialized over $\overline{H}_\pi$.

(4) $\overline{H}_\pi \otimes_{\mathbb{C}} \overline{H}_\pi \simeq \overline{H}_\pi \otimes_{\mathbb{C}} H_\pi$,

where $\overline{H}_\pi = H_\pi$ + H-module algebra structure

Example $H = H_q = \mathbb{C}\langle \sigma, \sigma^{-1}, \theta^{(1)} \rangle$ $\quad q\sigma\theta^{(1)} = \theta^{(1)}\sigma$

$\Delta(\sigma) = \sigma\otimes\sigma$, $\Delta(\sigma^{-1}) = \sigma^{-1}\otimes\sigma^{-1}$, $\Delta(\theta^{(1)}) = \theta^{(1)}\otimes 1 + \sigma\otimes\theta^{(1)}$

$\theta^{(n)} := \frac{1}{[n]_q!}(\theta^{(1)})^n$, $v_{m,n} := \sigma^m\theta^{(n)}$

$H = \bigoplus_{\substack{m\in\mathbb{Z}\\ n\in\mathbb{N}}} \mathbb{C} v_{m,n}$ $\quad$ CoMultiplication $\Delta(v_{m,n}) = \sum_{i+j=n} v_{m+j,i}\otimes v_{m,j}$

Representation $H_q = H \to M_2(\mathbb{C})$, $\sigma^{\pm1}\mapsto \begin{bmatrix} q & 0\\ 0 & 1\end{bmatrix}^{\pm1}$, $\theta^{(1)}\mapsto \begin{bmatrix} 0 & 0\\ 1 & 0\end{bmatrix}$

so that $q\sigma\theta^{(1)} = \theta^{(1)}\sigma$ for the images:

$$q\begin{bmatrix} q & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix} 0 & 0\\ 1 & 0\end{bmatrix} = \begin{bmatrix} 0 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix} q & 0\\ 0 & 1\end{bmatrix}.$$

Corepresentation matrix

$$\pi(V_{m,n}) = \pi(\sigma^m)\frac{1}{[m]_\xi!}\theta^{(m)} = \frac{1}{[m]_\xi!}\begin{pmatrix}\xi^m & 0\\ 0 & 1\end{pmatrix}\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}^n$$

$$= \begin{cases}\begin{bmatrix}\xi^m & 0\\ 0 & 1\end{bmatrix} & n=0 \\ \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} & n=1 \\ 0 & n \geq 2\end{cases}$$

Define $e : H_q \to \mathbb{C}$, $g \; H_q \to \mathbb{C}$ by

$$e(V_{mn}) = \begin{cases} q^m & n=0 \\ 0 & n\neq 0 \end{cases} \qquad g(V_{m,n}) = \begin{cases} 1, & n=1 \\ 0, & n\neq 1 \end{cases}$$

$$Y_\pi = \begin{bmatrix} e & 0 \\ g & 1 \end{bmatrix} \in M_2(H^\circ) \qquad \begin{aligned} \Delta(e) &= e \otimes e \\ \Delta(g) &= g \otimes e + 1 \otimes g \end{aligned} \qquad , \; eg = ge$$

$(\theta''') \subset H_q$ is a bilateral ideal and $H_q/(\theta''')$ is a Hopf algebra

$H_q \to H_q/(\theta''')$ is a morphism of Hopf algebra

$e;\ H_q \to$ $g\ H_q \to \mathbb{C}$ induce

$$\begin{tikzcd} H_q \arrow[r, "e"] \arrow[dr] & \mathbb{C} \\ & H_q/\theta''' \arrow[u, "\bar{e}"'] \end{tikzcd} \qquad \begin{tikzcd} H_q \arrow[rr, "g"] \arrow[dr] & & \mathbb{C} \\ & H_q/\theta''' \arrow[ur, "\bar{g}"'] & \end{tikzcd}$$

$\Delta(e)=e\otimes e$, $\Delta(e^{-1})=e^{-1}\otimes e^{-1}$, $\Delta(g)=1\otimes g+g\otimes e$

$eg=qge$

$H_\pi = \mathbb{C}\langle e, \ ^{-1}; g \rangle \supset$ smallest Hopf subalgebra $\subset H^\circ$ containing e, g

$=$