Transfer results in topological differential fields.

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Preliminaries-topological fields

Let K be a topological field of characteristic 0, namely a field with a topology compatible with the field operations, given for instance by an order $<$, (or several orders: $<_1, \cdots, <_e$) or by a valuation v, (or several valuations $v_1, \cdots, v_e$) or both...

We will consider the following cases:

- $(K, <)$ an ordered field
- (K, v) a non-trivially valued field
- (K, v) a valued p-adic field
- $(K, <, v)$ an ordered valued field
- $(K, <_1, \cdots, <_e)$ a field endowed with distinct e orderings.
- $(K, v_1, \cdots, v_e)$ a field endowed with distinct e valuations.

Preliminaries–topological differential fields

Now endow these fields with a derivation D, as freely as possible (i.e. no required interaction between D and the topology).

$\rightsquigarrow$ a notion of differential topological fields.

Under the assumption that the class of e.c. models (in the language without the derivation) is axiomatizable,

• How can we axiomatize the class of existentially closed (e.c.) expansions with a derivation?
• In these e.c. classes which properties transfer from their reducts?

Notation:

Let K be a differential field of characteristic 0 and let $K\{X\}$ be the ring of differential polynomials over K in one differential indeterminate X over K.

Let $f(X) \in K\{X\} \setminus K$, then we can write $f(X) = f^*(X, \ldots, X^{(n)})$ for some ordinary polynomial $f^*(X_0, \cdots, X_n) \in K[X_0, \cdots, X_n]$ and some natural number n that we choose minimal such; n is the order of f, denoted by $ord(f) = n$.

Write f as: $f = f_d \cdot {X^{(n)}}^d + \cdots + f_1 \cdot X^{(n)} + f_0$ where $f_0, \cdots, f_d \in K[X, X^{(1)}, \ldots X^{(n-1)}]$, $f_d \neq 0$ and $deg_{X^{(n)}} f$ is the degree of f in $X^{(n)}$.

The *separant* s_f of f is defined as $s_f = \frac{\partial f}{\partial X^{(n)}}$.

[Kolchin] The field K has a differential closure $\tilde{K} \models DCF_0$, namely a prime model extension (unique up to K-isomorphism) where one can solve any system of the form $f(X) = 0 \;\&\; g(X) \neq 0$, with $ord(g) < ord(f)$ for $f,\ g \in \tilde{K}\{X\}$.

Preliminaries–CODF

In the ordered case, M. Singer axiomatised the corresponding theory *CODF* of ordered differential existentially closed fields. The axiomatisation consists in asking to be able to solve any differential system of the form:
$f(x) = 0 \;\&\; \bigwedge_i g_i(x) > 0$, with $ord(g_i) \leq ord(f)$, whenever one can the solve the analogous algebraic system:
$f^*(\bar{x}) = 0 \;\&\; s_f^*(\bar{x}) \neq 0 \;\&\; \bigwedge_i g_i(\bar{x}) > 0$.

Later, Guzy and Rivière gave a geometric axiomatisation, modelled on the one for DCF_0 (Pierce-Pillay).

Observation (Singer): the isolated types in $S_1^{CODF}(\mathbb{Q})$ are not dense. Consider the clopen subset $[x^{(1)} = 1]$.
Observation: let $K \models CODF$, then in any cartesian product K^n, differential tuples $a^{\nabla} := (a, a^{(1)}, \cdots, a^{(n-1)})$ are dense.

[Singer] Given a model $\mathcal{U}$ of *CODF*, its differential closure is: $\mathcal{U}(i)$, $i^2 = -1$.

This relies on the following embedding theorem: if K is an ordered finitely generated differential field over $\mathbb{Q}$, then K is isomorphic to a field of real meromorphic functions in some neighbourhood of the origin.

Let $\mathcal{U}$ be an ordered field, model of *CODF* containing $\mathbb{R}(t)$, where t is a transcendental element and put on $\mathcal{U}(i)$ the product topology induced by the topology on $\mathcal{U}$.

• Consider the first Painlevé equation: $y^{(2)} = 6.y^2 + t$ in $\mathcal{U}(i)$. Then the set of solutions of that equation is dense in $\mathcal{U}(i)$.

Nullstellensatz

Let $A \subset K^n$, $I \subset K\{X_1, \cdots, X_n\}$. Then
$\mathcal{I}(A) := \{f \in K\{X_1, \cdots, X_n\} : \forall a \in A\ f(a) = 0\}$ and
$V(I) := \{\bar{a} \in K^n : \forall f \in I\ f(\bar{a}) = 0\}$.

An ideal $I \subset K\{X_1, \cdots, X_n\}$ is real if for any $u_1, \cdots, u_n \in K\{X_1, \cdots, X_n\}$ such that $\sum_i u_i^2 \in I$, then $u_i \in I$, for all i. Note that if I is real, I is radical.
An ideal I is differential if $a \in I$ implies that $D(a) \in I$.
Let $\mathcal{R}(I)$ be the smallest real ideal containing I.

[Brouette] (Nullstellensatz) Let I be a differential ideal of $K\{X_1, \cdots, X_n\}$, then $\mathcal{I}V(I) = \mathcal{R}(I)$.

Positivstellensatz

Note that there exists archimedean models of CODF (Michaux) and even the field of reals can be taken as the domain of a model of *CODF* (Brouette).

Let $K \models CODF$.
• Denote by $f_{n(k+1),d}(C,X) \in \mathbb{Z}\{C;X\}$ the general form of a polynomial of total degree $\leq d$ and order $\leq k$ belonging to $K\{X_1,\cdots,X_n\}$ with coefficients $C := (c_1,\cdots,c_m) \in K^m$, where $m = \binom{n(k+1)+d}{n(k+1)}$.

Let $S \subset K\{X_1,\cdots,X_n\}$ a finite set of differential polynomials s_i, $1 \leq i \leq e$, of order $\leq k$. Let $W_K(S) := \{\bar{x} \in K^n : \bigwedge_{i=1}^{e} s_i(\bar{x}) \geq 0\}$ and $W_K(S^*) := \{\bar{x} \in K^{n(k+1)} : \bigwedge_i s_i^*(\bar{x}) \geq 0\}$.

Positivstellensatz

[Brouette] Assume that there exists an open set O such that $O \subseteq W_K(S^*) \subseteq \bar{O}$.

• Let $f \in K\{X_1, \cdots, X_n\}$, then $f \restriction W \geq 0$ iff there exists $m \in \mathbb{N}$ such that $f.g =_W f^{2m} + h$, where g, $h \in P$, P the cone generated by $s_1, \cdots, s_e$.

• Effective version– (Prestel-Delzell) Assume that $\mathbb{R} \subset K$ and that $S \subset \mathbb{R}\{X_1, \cdots, X_n\}$. Assume further that $W_{\mathbb{R}}(S^*) \neq \emptyset$ and that for some $N \in \mathbb{N}$, we have $\forall \bar{x} \in W_{\mathbb{R}}(S^*)\ \|\bar{x}\| < N$. Then there exists a bound $b(n(k+1), e, d, N, S)$ such that $\forall \bar{x}$ [

$$(\forall \bar{z} \in W_K(S)\ \ \forall \bar{c}\ \|c\| < N\ \ f_{n(k+1),d}(\bar{c}, \bar{z}) > 2/N)\ \rightarrow$$

$$\exists \bar{y}\ (\|\bar{y}\|\ \leq b\ \&\ f_{n(k+1),d}(\bar{c}, \bar{x}) =$$

$$\sum_{\nu \in \{0,1\}^e} s_1^{\nu_1} \cdots s_e^{\nu_e} \sum_{i=1}^{\ell} f^{\nu i}_{n(k+1),b}(\bar{y}, \bar{x})^2)],$$

where $\ell = \binom{b+n(k+1)}{n(k+1)}$.

Scheme (DL)

Let $\mathcal{V}$ be a basis of neighbourhoods of 0.
We will express the fact that if a differential polynomial while considered as an ordinary algebraic polynomial has a zero then it has a differential zero *close* (relative to $\mathcal{V}$) to this algebraic zero.

Let $\langle L, \mathcal{V}\rangle$ be a differential topological $\mathcal{L}$-field. For each $n \in \omega^*$, let $\mathcal{V}_n$ be a basis of neighborhoods of $\bar{0} \in L^n$ in the product topology.

$\langle L, \mathcal{V}\rangle$ satisfies $(DL)_{\mathcal{V}}$ if for every $n \geq 1$, for every differential polynomial $f(X) = f^*(X, X^{(1)}, \ldots, X^{(n)})$ belonging to $L\{X\}$ and for every $W \in \mathcal{V}_n$, the following implication holds:
$(\exists \alpha_0, \ldots, \alpha_n \in L)(f^*(\alpha_0, \ldots, \alpha_n) = 0 \wedge s_f^*(\alpha_0, \ldots, \alpha_n) \neq 0) \Rightarrow$
$\Big((\exists z)(f(z) = 0 \wedge s_f(z) \neq 0 \wedge (z^{(0)} - \alpha_0, \ldots, z^{(n)} - \alpha_n) \in W)\Big).$

Building on previous work of M. Tressl:

[Guzy, P.] Any element of an inductive class $\mathcal{C}$ of differential topological $\mathcal{L}$-fields satisfying a property $(*)$ analogous to *largeness* can be embedded in another element of $\mathcal{C}$ satisfying a scheme (DL).

$(*)$ We assume that any element K of $\mathcal{C}$ has an extension in $\mathcal{C}$ which contains $K((t))$.

This allows us to axiomatize the class of e.c. elements of $\mathcal{C}$, in case the topology is first-order uniformly definable, given for instance by an order or a valuation.

Typically, we start with a universal theory T which has a model-completion T_c.

1. $T = OF \rightsquigarrow$(Tarski) $T_c = RCF$ o-minimal theory,
2. $T = VF_{0,0} \rightsquigarrow$(A. Robinson) $T_c = ACVF_{0,0}$ C-minimal minimal,
3. $T = OVF \rightsquigarrow$(G. Cherlin-M. Dickmann) $T_c = RCVF$ weakly o-minimal theory,
4. $T = p-$valued fields of p-rank d (Ax-Kochen-Ersov; Macintyre; Prestel-Roquette) $\rightsquigarrow$ $T_c =_p CF_d$ p-minimal theory. For instance, $pCF_1 := Th(\mathbb{Q}_p)$.

We will denote the expansion with a derivation by T_D and let $\mathcal{C}$ be the class of models of T_D.

• Then the class of existentially closed models in $\mathcal{C}$ is axiomatized by $T^*_{c,D} := T_{c,D} \cup (DL)$, provided $\mathcal{C}$ has property $(*)$.

We obtain for the theory $T^*_{c,D}$: if

1. $T_c = RCF$, CODF,
2. $T_c = ACVF_{0,0}$, an expansion of DCF_0,
3. $T_c = RCVF$, an expansion of CODF,

Transfer of properties from T_c to $T^*_{c,D}$

• [Guzy, P.] Under the above assumptions on T_c, the theory $T^*_{c,D} := T_{c,D} \cup (DL)$ is the model-completion of T_D and admits quantifier elimination.
• [N. Guzy, P.] The definable sets in models of $T^*_{c,D}$ can be endowed with a fibered dimension function.

Analogous result for *CODF*:
[Brihaye, Michaux, Rivière] Cell decomposition in models of *CODF* leading to a description of definable sets and the existence of such dimension function.

Instead of using a cell decomposition theorem, we used instead former results of van den Dries on Dimension of definable sets, algebraic boundedness and henselian fields.

[Michaux-Rivière] *CODF* is NIP.

We will prove an analogous result in our larger framework.

VC-dimension

Let $\phi(\bar{x};\bar{y})$ be a formula and $\mathcal{S}_\phi := \{\phi(M^m;\bar{b}) : \bar{b} \in M^n\}$.
A finite set A is shattered by ϕ if $|\{A \cap S : S \in \mathcal{S}_\phi\}| = 2^{|A|}$.
The VC-dimension of $\mathcal{S}_\phi$ is equal to the maximal cardinality of a finite set shattered by $\mathcal{S}_\phi$, if finite and $+\infty$, otherwise.
The VC-dimension of ϕ is the VC-dimension of the class $\mathcal{S}_\phi$.

Define $\pi_{\phi(\bar{x};\bar{y})}(t) := max_{A \subset M}|\{A \cap S : S \in \mathcal{S}_\phi, |A| = t\}|$.
[Sauer, Shelah] If the VC-dimension of a set is finite, then the function sending t to $\pi_{\phi(\bar{x};\bar{y})}(t)$ is bounded by a polynomial of t.

The VC-density of ϕ is equal to the infimum of all real numbers r such that $\frac{\pi_{\phi(\bar{x};\bar{y})}(t)}{t^r}$ is bounded (as a function of t).

Given a formula $\phi(\bar{x};\bar{y})$, let $\phi^d(\bar{x};\bar{y})$ be the dual formula, namely $\phi(\bar{y};\bar{x})$.

Let $S_\phi(B)$ be maximal consistent sets of formulas of the form $\{\phi(\bar{x};\bar{b}) : b \in B'\} \cup \{\neg\phi(\bar{x};\bar{b}) : \bar{b} \in B - B'\}$.

If $B' \subseteq B$ of B is cut out by $\phi^d(\bar{x};\bar{y})$, then we send it to the maximal consistent sets of formulas of the form $\{\phi(\bar{x};\bar{b}) : b \in B'\} \cup \{\neg\phi(\bar{x};\bar{b}) : \bar{b} \in B - B'\}$.

Set $\pi_{\phi^d(\bar{x};\bar{y})}(t) := max\{|S_\phi(B)| : B \subset M^n, |B| = t\}$.

The dual VC-density of ϕ is equal to the infimum of all real numbers r such that $\frac{\pi_{\phi^d}(t)}{t^r}$ is bounded (as a function of t).

Non independance property

A formula is NIP if its VC-dimension is finite. A theory is NIP if all its formulas are NIP.

- [Guzy, P.] The NIP property transfers from T_c to $T^*_{c,D}$.

So we can apply this to the theories: RCF, $ACVF_{0,0}$, $RCVF$, ${}_pCF_d$.

Together with the fact that $T^*_{c,D}$ admits quantifier elimination (q.e.), it is a corollary of the following observation.

Let T be a model-complete theory and let $\mathcal{M} \models T$. Assume that $\mathcal{M}$ can be embedded into an $\mathcal{L}_D$-structure $\mathcal{M}^*$ whose $\mathcal{L}$-reduct is a model of T and which has an open $\mathcal{L}_D$ formula ϕ with the independence property. Then the open $\mathcal{L}$-formula ϕ^* has the independence property in $\mathcal{M}$.

NTP_2-case

• When $T = OF_e$, we get by a result of van den Dries a model-companion $T_c = \overline{OF_e}$ (maximal PRC_e fields).

• In this case the axiomatisation of the e.c. class of the expansion by a derivation can be axiomatized by $\overline{OF_{e,D}}^{\omega}$.

• In this case, the theory $\overline{OF_e}$ is not longer NIP (Duret), but one has the other *good* combinatorial property: NTP_2, by a result of S. Montenegro (for a larger class: PRC_e).

[Montenegro] PRC_e is NTP_2.

Claim: This property transfers to $\overline{OF_{e,D}}^{\omega}$.

VC-density

Theorem (Aschenbrenner-Dolich-Haskell-Macpherson-Starchenko)

Let T be a weakly o-minimal theory. Then, the dual VC-density of a formula $\phi(\bar{x}; \bar{y})$ is bounded by $|\bar{x}|$.

For instance, $T = RCF$ or $T = RCVF$.
Also, one can derive a result on $ACVF_{0,0}$ by interpretation.

Theorem (Aschenbrenner-Dolich-Haskell-Macpherson-Starchenko)

Let $T = Th(\mathbb{Q}_p)$. Then, the dual VC-density of a formula $\phi(\bar{x}; \bar{y})$ is bounded by $2.|\bar{x}| - 1$.

Question: What can we say about the dual VC-density of their differential field expansions?

Cell Decomposition Property

Let $\phi(x_1, \cdots, x_n)$ be a quantifier-free $\mathcal{L}_D$-formula, for each x_i, $1 \leq i \leq n$, let m_i be the maximal natural number m such that $x_i^{(m)}$ occurs in an atomic subformula.
Then, we denote by $\phi^*((x_{i,j})_{i=1,j=0}^{n,m_i})$ the formula we obtain from ϕ by replacing each $x_i^{(j)}$ by $x_{i,j}$.

Let $\mathcal{M}$ be a topological $\mathcal{L}$-field and $S = \phi(M)$, then we denote by $S^{alg} := \phi^*(M)$.

• [Projection maps]
$\pi_{(i_1,\cdots,i_k)} : M^n \to M^k : (x_1, \cdots, x_n) \to (x_{i_1}, \cdots, x_{i_k})$.
$\pi_k : M^n \to M^k : (x_1, \cdots, x_n) \to (x_1, \cdots, x_k)$.

• A cell consists either of a point in M^n and is of dimension 0, or is a definable subset X of M^n such that there exist $1 \leq k \leq n$ and k positive integers $1 \leq i_1 < \cdots < i_k \leq n$ such that the projection map $\pi^n_{(i_1,\cdots,i_k)}(X)$ is a definable homeomorphism and the image is open in M^k. Then $\mathcal{L}$-dim(X)=k.

Cell Decomposition Property

• The $\mathcal{L}$-structure $\mathcal{M}$ has the cell decomposition property (CDP) if for any A-definable subset $X \subset M^n$, $A \subset M$, can be partitioned into finitely many cells and if given any A-definable function f from X to $\mathcal{M}$ there exist a partition of X into finitely many A-definable cells X_i such that $f\ |_{X_i}$ is continuous.

• [Mathews] Further assume that T_c has finite Skolem functions and the local continuity property of zeroes of polynomials, then in any model of T_c, we have the cell decomposition property.

This will apply to $T_c = RCF$, or $RCVF$ or ${}_pCVF_d$

Definable sets in models of $T^*_{c,D}$

Lemma (P.)

Let $\mathcal{M} \models T^*_{c,D}$ and K a differential subfield of M. Given an $\mathcal{L}_{D,K}$-definable set S in $\mathcal{M}$, there exists an $\mathcal{L}_K$-definable subset S^* of S^{alg} such that S is included and dense $\pi_1(S^*)$. Moreover, S^* can be partionned into a finite union of $\mathcal{L}_K$-definable cells $\tilde{C}$ such that the $\mathcal{L}$-generic tuples of the form a^{∇}, $a \in S$, are dense in $\pi_d(\tilde{C})$, for some d, $\tilde{C} \subset dcl_{\mathcal{L}_K}(\pi_d(\tilde{C}))$ and $d = \mathcal{L} - dim(\pi_d(\tilde{C})) = \mathcal{L} - dim(\tilde{C})$.

Lemma (P.)

Moreover, to any $\mathcal{L}_{D,K}$-definable unary function f with domain $S \subset M$, we can associate a $\mathcal{L}_K$-definable function f^* defined on $S^* \subset S^{alg}$ such that f and f^* coincide on a dense subset of S. Moreover, for each $\tilde{C} \in \mathcal{P}_S^*$ of $\mathcal{L}$-dimension d the projection $\pi_d(\tilde{C})$ can be partitioned in a finite union of $\mathcal{L}_K$-definable cells C such that $f^* \restriction C$ is continuous.

This generalises for $\mathcal{L}_D$-definable set $S \subset M^k$.

VC-density back in CODF

Let $\mathcal{M} \models CODF$, assume that $\mathcal{M}$ sufficiently saturated and let $\phi(x; \bar{y})$ be an $\mathcal{L}$-formula. Let $K \subset M$ and $S = \phi(M, \bar{k})$. Let $\phi^*_{mod}(x, \cdots, x^{(m)}; \bar{y}, \cdots, \bar{y}^{(n)})$ be the formula constructed above, namely such that $S^* = \phi^*_{mod}(M; \bar{k}, \cdots, \bar{k}^{(n)})$.

Proposition

The dual VC-density of the $\mathcal{L}_D$-formula $\phi(x; \bar{y})$ is equal to the dual VC-density of $\phi^*_{mod}(x, \cdots, x^{(m)}; \bar{y}, \cdots, \bar{y}^{(n)})$.

Elimination of Imaginaries

Using the above lemmas and the fact that in o-minimal theories, cells are definably connected, we show transfer of elimination of imaginaries from RCF to $CODF$.

Proposition (P.)

Let $\mathcal{K}$ be a model of $CODF$. Given an $\mathcal{L}_D$-definable set $S \subset K^n$ with $S = \phi(K, \bar{a})$, there exists an $\mathcal{L}_D$-formula $\psi(\bar{x}, \bar{y})$ and a unique tuple $\bar{b} \subset K$ such that $\mathcal{K} \models \forall \bar{x}\ \phi(\bar{x}, \bar{a}) \leftrightarrow \psi(\bar{x}, \bar{b})$.

Definable types

Now we want to see under which circumstances the induced structure on a definable set by the ambient one is already present. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, C a definable subset of $\mathcal{M}$ with $dcl(C) = C$, $\bar{a} \in M^n$ and $\phi(x, \bar{a})$ an $\mathcal{L}$-formula.

The type of $\bar{a}$ in $\mathcal{M}$ over C is C-definable if for any $\mathcal{L}$-formula $\psi(\bar{x}, \bar{y})$ there exists an C-formula $d\psi(\bar{y})$, such that for any $\bar{c} \in C$, $\mathcal{M} \models \psi(\bar{a}, \bar{c})$ iff $\mathcal{M} \models d\psi(\bar{c})$.

So if the type of $\bar{a}$ in $\mathcal{M}$ over C is C-definable, then the definable subset of C: $\phi(C, \bar{a})$ is definable by a C-formula $d\psi(C)$.

Since we are working in a non-stable context, we know that not all types are definable. But one has a description of the definable ones over models in the following cases (for instance):

1. *RCF* van den Dries/o-minimal case: Marker-Steinhorn; Pillay,
2. ${}_pCF_d$ Delon, Bélair,
3. *RCVF* Mellor,
4. *ACVF* ...../Cubides-Delon.

In CODF

Let $K_0 \subset K_1$ be two ordered fields. A cut of K_0 is a partial type of the form $C_1 < v < C_2$ with $C_1 \cup C_2 = K_0,\ C_1,\ C_2 \neq \emptyset$ and $max(C_1) \notin K_0$, $min(C_2) \notin K_0$.
Recall that K_0 is Dedekind complete in K_1 if no cut in K_0 is realised in K_1.

• (Marker-Steinhorn) Let T be an o-minimal theory and let $p(\bar{x}) \in S_n^T(M)$, where $\mathcal{M} \models T$. Then $p(\bar{x})$ is definable over M iff $\mathcal{M}$ is Dedekind complete in any $\mathcal{M}(\bar{a})$, where $\mathcal{M}(\bar{a})$ is the prime model generated by $\mathcal{M}$ and $\bar{a}$ a realisation of $p(\bar{x})$.

• We need that the type of a *small element* is definable (over a model).

Let $\mathcal{K} \models CODF$ and let A be differential subfield of K.
• Let $p(x) \in S_1^{CODF}(A)$ and assume that $A \models RCF$. Then $p(x)$ is definable over A if and only if A is Dedekind complete in $A(v, \cdots, v^{(n)}, \cdots; n \in \omega)$, for any element v realizing $p(x)$.

Proposition (Brouette, P.)

Let $A \models RCF$. Then the definable types are dense in $S_1^{CODF}(A)$.

One uses the axiomatisation of CODF (and its proof) + the fact that the type of 0^+ is definable over a model of RCF.

One can extend this result to $ACVF_{0,0}$, ${}_pCF_d$ and $RCVF$.

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