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Double-angle formulae and algebraic independence

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1 Double-angle formulae

$$\cos 2x = 2(\cos x)^2 - 1,$$

$$\mathrm{e}^{2x} = (\mathrm{e}^x)^2,$$

$$\wp(2x) = \frac{1}{16} \cdot \frac{16\wp(x)^4 + \cdots}{4\wp(x)^3 - \cdots}$$

$$(g_2^3 - 27g_3^2 \neq 0),$$

$$2x = 2 \cdot x.$$

2 Th. (alg. indep.)

$f_1(x), \ldots, f_n(x)$: non-const. merom. func.,

$$f_i(2x) = A_i(f_i(x))/B_i(f_i(x)),$$

where $A_i, B_i \in \mathbb{C}[X]$, $(A_i, B_i) = 1$.

$$c_i := \max\{\deg A_i, \deg B_i\}.$$

If $c_i \neq c_j$ $(i \neq j)$, then $f_1, \ldots, f_n$ are alg. indep./$\mathbb{C}$.

3 Cor.

x, e^x and $\wp(x)$ are alg. indep./$\mathbb{C}$.

$$2x = 2 \cdot x, \qquad \boxed{1}$$

$$\mathrm{e}^{2x} = (\mathrm{e}^x)^2, \qquad \boxed{2}$$

$$\wp(2x) = \frac{1}{16} \cdot \frac{16\wp(x)^4 + \cdots}{4\wp(x)^3 - \cdots} \qquad \boxed{4}$$

$$(g_2^3 - 27g_3^2 \neq 0).$$

4 Proof of Th.

The functions $f_1, \ldots, f_n$ are non-const.

$$[\mathbb{C}(f_i(x)) : \mathbb{C}(f_i(2x))] = c_i$$

from

$$f_i(2x) = \frac{A_i(f_i(x))}{B_i(f_i(x))}, \ (A_i, B_i) = 1,$$

$$c_i = \max\{\deg A_i, \deg B_i\}.$$

Hence $[\mathbb{C}(f_i(x)) : \mathbb{C}(f_i(2^k x))] = c_i^k$.

Proof of Th. by induction on n.

$$F(x) := \{f_1(x), \ldots, f_n(x)\},$$

$$F^{(p)}(x) := \{f_i(x) : i \neq p\},$$

$$D_k := [\mathbb{C}(F(x)) : \mathbb{C}(F(2^k x))],$$

$$D_k^{(p)} := [\mathbb{C}(F^{(p)}(x)) : \mathbb{C}(F^{(p)}(2^k x))].$$

By the induc. hypothesis, $f_1, \ldots, f_n$ excluding f_p are alg. indep./$\mathbb{C}$. Hence, by $F^{(p)}(x) = \{f_i(x) : i \neq p\}$,

$$\begin{aligned} D_k^{(p)} &= [\mathbb{C}(F^{(p)}(x)) : \mathbb{C}(F^{(p)}(2^k x))] \\ &= \prod_{i \neq p} [\mathbb{C}(f_i(x)) : \mathbb{C}(f_i(2^k x))] \\ &= \prod_{i \neq p} c_i^k = \left(\prod_i c_i^k \right) c_p^{-k}. \end{aligned}$$

Assume $f_1, \ldots, f_n$ are alg. dep./$\mathbb{C}$.
f_p is algebraic/$\mathbb{C}(F^{(p)}(x))$.

$$\begin{aligned} d^{(p)} &:= [\mathbb{C}(F(x)) : \mathbb{C}(F^{(p)}(x))] \\ &= [\mathbb{C}(F^{(p)}(x), f_p) : \mathbb{C}(F^{(p)}(x))] \\ &< \infty, \\ d_k^{(p)} &:= [\mathbb{C}(F(2^k x)) : \mathbb{C}(F^{(p)}(2^k x))] \\ &\leq d^{(p)}. \end{aligned}$$

$$
\begin{aligned}
&[\mathbb{C}(F(x)) : \mathbb{C}(F^{(p)}(2^k x))] \\
&= d^{(p)} D_k^{(p)} = D_k d_k^{(p)}
\end{aligned}
$$

by

$$
\begin{aligned}
D_k &= [\mathbb{C}(F(x)) : \mathbb{C}(F(2^k x))], \\
D_k^{(p)} &= [\mathbb{C}(F^{(p)}(x)) : \mathbb{C}(F^{(p)}(2^k x))], \\
d^{(p)} &= [\mathbb{C}(F(x)) : \mathbb{C}(F^{(p)}(x))], \\
d_k^{(p)} &= [\mathbb{C}(F(2^k x)) : \mathbb{C}(F^{(p)}(2^k x))].
\end{aligned}
$$

Choose $1 \le p, p' \le n$ s.t. $p \neq p'$ and $c_p < c_{p'}$.

$$\left(\frac{c_{p'}}{c_p}\right)^k = \frac{D_k^{(p)}}{D_k^{(p')}} \qquad \boxed{D_k^{(p)} = c_p^{-k} \prod_i c_i^k}$$

$$= \frac{D_k d_k^{(p)}}{d^{(p)}} \cdot \frac{d^{(p')}}{D_k d_k^{(p')}} \le \frac{d^{(p)} d^{(p')}}{d^{(p)} \cdot 1}$$

$= d^{(p')}$, a contradiction. QED.

5 Th. (generalized)

$f_1(z), \ldots, f_n(z) \in \mathbb{C}((z)) \setminus \mathbb{C}$:

$$f_i(\tau z) = A_i(f_i(z))/B_i(f_i(z)),$$

where $\tau z \in \mathbb{C}[[z]] \setminus \mathbb{C}$ (ex. qz, z^d),
$A_i, B_i \in \mathbb{C}[X]$, $(A_i, B_i) = 1$.

$$c_i := \max\{\deg A_i, \deg B_i\}.$$

If $c_i \neq c_j$ $(i \neq j)$, then $f_1, \ldots, f_n$ are alg. indep./$\mathbb{C}$.

6 More general result

Nishioka, S.,
Algebraic independence of solutions of first-order rational difference equations,
Results in Mathematics, Volume 64, Issue 3 (2013), 423–433.
doi:10.1007/s00025-013-0324-8

In terms of difference algebra.

7 Transcendental num.

K : an algebraic num. field.

$d \in \mathbb{Z}$: $d > 1$.

$f \in K[[z]]$ with radius of conv. $R > 0$:

$$f(z^d) = A(f(z))/B(f(z)),$$

where $A, B \in K[z][X]$, $(A, B) = 1$.

$$m := \max\{\deg_X A, \deg_X B\}.$$

$\Delta(z)$: the resultant of A and B.

8 Mahler's theorem

(K. Mahler, 1929) Suppose $m < d$, $f(z)$ is trans./$K(z)$. If $\alpha \in \overline{\mathbb{Q}}$ satisfies

$$0 < |\alpha| < \min\{1, R\},$$

$$\Delta(\alpha^{d^k}) \neq 0 \ (k \geq 0),$$

then $f(\alpha)$ is a transcendental num.

(Ku. Nishioka, 1982) The above th. still holds when $m < d^2$.

9 Existence

$$f(z^d) = \frac{a_d f(z)^d + \cdots + a_m f(z)^m}{1 + b_1 f(z) + \cdots + b_m f(z)^m},$$

where $a_i, b_i \in K$, $a_m \neq 0$ or $b_m \neq 0$, $a_d \neq 0$,

$$(\cdots + a_m X^m, \cdots + b_m X^m) = 1.$$

There is a sol. $f(z) \in K''\{\{z\}\} \setminus K''$, where K'' is a finite extention of K (cf. K. Mahler, 1983)

10 Transcendence

$f(z) \in K''\{\{z\}\} \setminus K''$ (the above) :

$$f(z^d) = \frac{a_d f(z)^d + \cdots + a_m f(z)^m}{1 + b_1 f(z) + \cdots + b_m f(z)^m}.$$

z (the independent var.) : $z^d = (z)^d$.

By Th. (generalized), if $m \neq d$, then $f(z)$ and z are alg. indep./$\mathbb{C}$, and so $f(z)$ is trans./$\mathbb{C}(z)$.

11 Th. (trans. num.)

$f(z) \in K''\{\{z\}\} \setminus K''$ (the above) :

$$f(z^d) = \frac{a_d f(z)^d + \cdots + a_m f(z)^m}{1 + b_1 f(z) + \cdots + b_m f(z)^m}.$$

Suppose $d < m < d^2$.
$R > 0$: the radius of convergence.
If $\alpha \in \overline{\mathbb{Q}}$ satisfies $0 < |\alpha| < \min\{1, R\}$,
then $f(\alpha)$ is a transcendental num.

12 End

Thank you.