# Model theory of pseudo real closed fields.

Samaria Montenegro

Université Paris 7

Théorie des modèles, équations différentielles
et aux différences et applications
CIRM, 7 au 10 avril 2015

# Pseudo algebraically closed fields (PAC fields )

**Regular extensions:** Let $M$ and $N$ be fields of characteristic 0 such that $M \subseteq N$. We say that $N$ is a *regular extension* of $M$ if $N \cap M^{alg} = M$.

**Definition (Ax)**

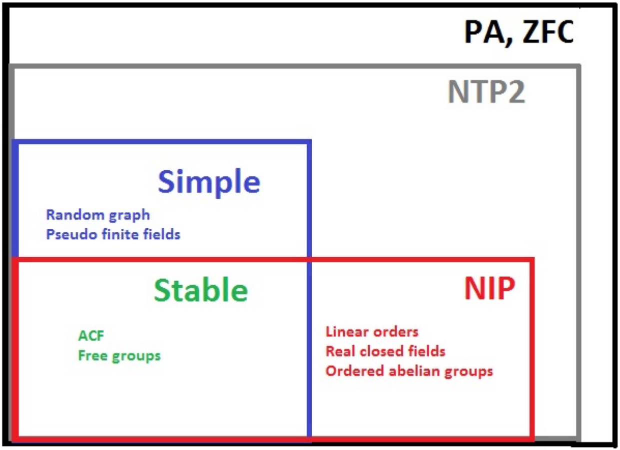
*A PAC field is a field $M$ such that $M$ is existentially closed (in $\mathcal{L}_{\mathcal{R}}$ the language of rings) into each regular field extension of $M$.*

The algebraically closed fields and the pseudo finite fields are examples of PAC fields.

The class of PAC fields is axiomatizable in $\mathcal{L}_{\mathcal{R}}$.

# Shelah's classification theory

Shelah classified complete first-order theories by their ability to encode certain combinatorial configurations.



# PAC and their stability theoretic properties

**Fact (Duret)**

*If $M$ is a PAC field which is not an algebraically closed field, then $Th_{\mathcal{L}_{\mathcal{R}}}(M)$ is not NIP.*

**Bounded fields:** A field $M$ is called bounded if for any integer $n$, $M$ has only finitely many extensions of degree $n$.

**Fact**

*Let $M$ be a PAC field, then:*

1. *[Chatzidakis-Pillay] If $M$ is bounded, then $Th_{\mathcal{L}_{\mathcal{R}}}(M)$ is simple.*
2. *[Chatzidakis] Si $Th_{\mathcal{L}_{\mathcal{R}}}(M)$ is simple, then $M$ is bounded.*

# Pseudo real closed fields (PRC fields )

This notion of PAC field has been generalized by Basarab and Prestel for ordered fields.

**Definition**

*A field $M$ is called PRC if $M$ is existentially closed (respect to $\mathcal{L}_{\mathcal{R}}$) into each regular field extension $N$ to which all orderings of $M$ extend.*

The PAC fields and the real closed fields are examples of PRC fields.

[Prestel] The class of PRC fields is axiomatizable in $\mathcal{L}_{\mathcal{R}}$.

# Pseudo real closed fields (*PRC* fields )

**Fact (Prestel)**

*Let $M$ be a PRC field, then:*

1. *If $<$ is an order on $M$, then $M$ is dense in $(\overline{M}^r, \overline{<}^r)$ (the real closure of $M$ with respect to $<$).*
2. *If $<_i$ and $<_j$ are different orders on $M$, then $<_i$ and $<_j$ induce different topologies.*

# PRC fields and the independence property

**Theorem**

*Let $M$ be a PRC field which is neither algebraically closed nor real closed. Then $Th(M)$ has the independence property.*

***Proof***: *Prestel showed that algebraic extensions of PRC fields are PRC. Then $M(\sqrt{-1}))$ is PRC. Since $M(\sqrt{-1}))$ has no orderings then it is a PAC field. By Duret $M(\sqrt{-1}))$ has the independence property. As it is interpretable in $M$, then $M$ has the independence property.*

A. Chernikov, I. Kaplan and P. Simon conjectured the following:

Let $M$ be a PRC field. Then $M$ is bounded if and only if $Th(M)$ is $NTP_2$.

# $NTP_2$ theories

**Definition**

*Fix $\mathcal{L}$ a language and $T$ a complete $\mathcal{L}$-theory. We work inside a monster model $\mathbb{M}$ of $T$.*
*We say that $\phi(\bar{x}, \bar{y})$ has $TP_2$ if there are $(a_{lj})_{l,j<\omega} \in \mathbb{M}^{|y|}$ and $k \in \omega$ such that:*

1. $\{\phi(\bar{x}, a_{lj})_{j\in\omega}\}$ *is $k$-inconsistent for all $l < \omega$.*
2. *For all $f : \omega \to \omega, \{\phi(\bar{x}, a_{lf(l)}) : l \in \omega\}$ is consistent.*

A theory is called $NTP_2$ if no formula has $TP_2$.

### Theorem

*If $M$ is an unbounded PRC field, then $Th(M)$ is not $NTP_2$.*

### Lemma

*Let $M$ be an unbounded PAC field. Then $Th(M)$ is not $NTP_2$.*

**Proof of theorem:** Let $M$ be an unbounded PRC field. Then $M(\sqrt{-1})$ is a unbounded PAC field and it is interpretable in $M$. This implies by Lemma that $M$ is not $NTP_2$.

# Pseudo real closed fields (PRC fields )

**Notation:** We fix a bounded PRC field $M$, which is not real closed, and $M_0 \prec M$. Let $\mathcal{L} := \mathcal{L}_{\mathcal{R}} \cup \{c_m : m \in M_0\}$. The boundedness condition implies that $M$ has only finitely many orders. Let $\{<_1, \ldots, <_n\}$ be the orders on $M$. If $n = 0$ $M$ is a PAC field. We will suppose that $n \geq 1$. Let $\mathcal{L}_n := \mathcal{L} \cup \{<_1, \ldots, <_n\}$ and $\mathcal{L}^{(i)} := \mathcal{L} \cup \{<_i\}$.
Let $T := Th_{\mathcal{L}_n}(M)$. Denote by $M^{(i)}$ a fixed real closure of $M$ with respect to $<_i$.

**Lemma**

*For all $i \in \{1, \ldots, n\}$ we can define the order $<_i$ by an existential $\mathcal{L}$-formula.*

# Definable 1-sets

**Definition**

1. *A subset of $M$ of the form $I = \bigcap_{i=1}^{n}(I^i \cap M)$ with $I^i$ a non-empty $<_i$-open interval in $M^{(i)}$ is called a multi-interval.*
2. *A definable subset $S$ of a multi-interval $I = \bigcap_{i=1}^{n}(I^i \cap M)$ is called multi-dense in $I$ if for any multi-interval $J \subseteq I$, $J \cap S \neq \emptyset$*

**Remark:** Every multi-interval is non empty.
**AT** Let $\tau_1, \ldots, \tau_n$ different topologies on a field $F$ induced by orders or valuations. For each $i \in \{1, .., n\}$, let $U_i$ be a non-empty $\tau_i$-open subset of $M$. Then $\bigcap_{i=1}^{n} U_i \neq \emptyset$.

# Density Theorem

## Theorem

*Let $\phi(x,\bar{y})$ be an $\mathcal{L}_n$-formula and let $\bar{a}$ be a tuple in $M$. Then there are a finite set $A \subseteq \phi(M,\bar{a})$, $m \in \mathbb{N}$ and $I_1, \ldots, I_m$, with*

$I_j = \bigcap_{i=1}^{n} (I_j^i \cap M)$ *a multi-interval such that:*

1. $A \subseteq acl(\bar{a})$,
2. $\phi(M,\bar{a}) \subseteq \bigcup_{j=1}^{m} I_j \cup A$,
3. $\{x \in I_j : M \models \phi(x,\bar{a})\}$ *is multi-dense in* $I_j$ *for all* $1 \leq j \leq m$,
4. *the set* $I_j^i \cap M$ *is definable in* $M$ *by a quantifier-free* $\mathcal{L}^{(i)}(\bar{a})$*-formula, for all* $1 \leq j \leq m$ *and* $1 \leq i \leq n$.

# Amalgamation Theorem:

**Theorem**

*Let* $E = acl(E) \subseteq M$ *and* $a_1, a_2, d$ *tuples of* $M$ *such that:* $d$ *is ACF-independent of* $\{a_1, a_2\}$ *over* $E$, $tp_{\mathcal{L}}(a_1/E) = tp_{\mathcal{L}}(a_2/E)$, *and* $qftp_{\mathcal{L}_n}(d, a_1/E) = qftp_{\mathcal{L}_n}(d, a_2/E)$, *where* $\mathcal{L}_n = \mathcal{L} \cup \{<_1, \ldots, <_n\}$.
*Suppose that* $E(a_1)^{alg} \cap E(a_2)^{alg} = E^{alg}$.
*Then there exists a tuple* $d^*$ *in some elementary extension* $M^*$ *of* $M$ *such that:*

1. $d^*$ *is ACF-independent of* $\{a_1, a_2\}$ *over* $E$,
2. $tp_{\mathcal{L}}(d^*, a_1/E) = tp_{\mathcal{L}}(d^*, a_2/E)$,
3. $tp_{\mathcal{L}}(d^*, a_1/E) = tp_{\mathcal{L}}(d, a_1/E)$.

# Strong

## Definition

*Let $p(x)$ be a (partial) type. An inp-pattern of depth $\lambda$ in $p(x)$ consists of $(\bar{a}_l, \phi_l(x, y_l), k_l)_{l<\lambda}$ with $\bar{a}_l = (a_{lj})_{j\in\omega}$ and $k_l \in \omega$ such that:*

1. $\{\phi_l(x, a_{l,j})\}_{j<\omega}$ *is $k_l$-inconsistent, for each $l < \lambda$.*
2. $\{\phi_l(x, a_{l,f(l)})\}_{l<\lambda} \cup p(x)$ *is consistent, for any $f : \lambda \to \omega$.*

*The burden of a partial type $p(x)$ is the supremum of the depths of inp-patterns in it. We denote the burden of $p$ by $bdn(p)$.*

## Definition

*$T$ is called strong if there is no inp-pattern of infinite depth on it. Clearly, if $T$ is strong then it is $NTP_2$.*

# Strong

**Theorem**

*Let $M$ be a bounded PRC field with exactly $n$ orders. Then $Th(M)$ is strong and $bdn(x = x) = n$.*

**Idea of the proof:** If $M$ is real closed, then $Th_{\mathcal{L}_{\mathcal{R}}}(M)$ is strong of burden 1. We will suppose that $M$ is not real closed.
$bdn(x = x) \geq n$ : For $l \in \{0, \ldots, n-1\}$, define the formula $\varphi_l(x, y) := y <_{l+1} x <_{l+1} y + 1$. Take $((a_{l,j})_{j\in\omega})_{l\leq n-1}$, such that $a_{l,j+1} = a_{l,j} + 1$. Using the Approximation Theorem we have that $(\bar{a}_l, \varphi_l(x, y), 2)_{0\leq l<n}$, with $\bar{a}_l = (a_{l,j})_{j\in\omega}$ is an inp-pattern of depth $n$.

# Strong

$bdn(x = x) \leq n$ : Suppose by contradiction that there is an inp-pattern $(\bar{a}_l, \phi_l(x, y_l), k_l)_{0 \leq l < n+1}$ of depth $n + 1$;

**Step 1:**

By compactness we can take $\bar{a}_l := (a_{l,j})_{j \in \kappa}$, with $\kappa$ a sufficiently large cardinal.
We can suppose [Chernikov] that $|x| = 1$ and that the array $(\bar{a}_l)_{l < n+1}$ has rows mutually indiscernible over $E$.

# Strong

**Step 2:** Using density theorem, indiscernibility and properties of inp-patterns [Chernikov] we can suppose that for all

$0 \leq l < n+1, j < \kappa$ there is $I_{l,j} = \bigcap_{i=1}^{n} (I^i_{l,j} \cap M)$ a multi-interval

such that:

1. $\phi_l(M, a_{lj}) \subseteq I_{l,j}$
2. $\{x \in I_{l,j} : M \models \phi_l(x, a_{lj})\}$ is multi-dense in $I_{l,j}$.
3. $I^i_{l,j} \cap M$ is definable in $M$ by a quantifier-free $\mathcal{L}^{(i)}(E(a_{l,j}))$-formula $\theta_i(x, a_{l,j})$, where $\mathcal{L}^{(i)} = \mathcal{L} \cup \{<_i\}$.

# Strong

**Step 3:** There exists $0 \leq l \leq n$ such that $\bigcap_{j \in \kappa} I^i_{l,j} \neq \emptyset$, for all $1 \leq i \leq n$.

Using the fact that $M$ is $<_i$-dense in $M^{(i)}$ for all $i \in \{1, \ldots, n\}$ and saturation we can find for all $i \in \{1, \ldots n\}$, an $<_i$-interval $I^i$ such that:

1. $I^i \subseteq \bigcap_{j < \kappa} I^i_{lj}$,
2. $I^i = (c_i, d_i)_i$ with $c_i, d_i \in M$.

# Strong

**Step 4:** Using Erdös-Rado and compactness we can suppose that there is a countable subsequence of $(a_{l,j})_{j\in\kappa}$ which is indiscernible over $acl(E(c_i, d_i)_{i\leq n})$.
We replace $E$ by $acl(E(c_i, d_i)_{i\leq n})$ and then we can suppose that $I^i$ is definable in $M$ with parameters in $E$.

Denote by $I := \bigcap_{i=1}^{n}(I^i \cap M)$.

**Remark:** For all $l, j$, $\{x \in I : M \models \phi_l(x, a_{lj})\}$ is multi-dense in $I$

# Strong

**Step 5**:

**Lemma**

*Let $E = acl(E) \subset M$ and $(a_j)_{j\in\omega}$ an indiscernible sequence over $E$. Let $\phi(x, \bar{y})$ be an $\mathcal{L}(E)$-formula and $I$ a multi-interval definable over $E$ such that $\{x \in I \cap M : M \models \phi(x, a_0)\}$ is multi-dense in $I$. Then $p(x) := \{\phi(x, a_j)\}_{j\in\omega}$ is consistent.*

This Lemma implies that $\{\phi_I(x, a_{I,j})\}_{j\in\omega}$ is consistent.
This contradicts the $k_I$-inconsistency.

# Burden of types

**Theorem**

*Let $n \geq 1$ and let $M$ be a bounded PRC field with exactly $n$ orders. Let $r \in \mathbb{N}$ and $\bar{a} := (a_1, \ldots, a_r) \in M^r$. Then $bdn(\bar{a}/A) = n \cdot trdeg(A(\bar{a})/A)$. Therefore the burden is additive (i.e. $bdn(\bar{a}\bar{b}/A) = bdn(\bar{a}/A) + bdn(\bar{b}/A\bar{a})$).*

# Forking and dividing

## Definition

Let $T$ be a theory $T$ and $\mathbb{M}$ a monster model of $T$. Let $A \subseteq \mathbb{M}$ and let $a$ be a tuple of $\mathbb{M}$.

1. *The formula* $\psi(x, a)$ *divides over A* if there exists $k \in \mathbb{N}$ and an indiscernible sequence over $A$, $(a_j)_{j\in\omega}$ such that: $a_0 = a$ and $\{\psi(x, a_j) : j \in \omega\}$ is $k$-inconsistent.
2. *The formula* $\phi(x, a)$ *forks over A* if there is a number $m \in \mathbb{N}$ and formulas $\psi_j(x, a_j)$ for $j < m$ such that $\phi(x, a) \vdash \bigvee_{j<m} \psi_j(x, a_j)$ and $\psi_j(x, a_j)$ divides over $A$ for every $j < m$.
3. A *type p forks (divides) over A* if there is a formula from $p$ which forks (divides) over $A$.

# Forking and dividing

## Definition

Let $A \subseteq \mathbb{M}$. We say that $A$ is an *extension base* if for all tuple $a$ in $M$, $tp(a/A)$ does not fork over $A$. Denote by $a \mathop{\smile\!\!\!\!\!\!|}_A b$ if $tp(a/Ab)$ does not fork over $A$.

## Theorem (Chernikov-Kaplan)

*If $T$ is $NTP_2$ and all sets are extensions bases, then forking equals dividing.*

# Forking and dividing in PRC fields

Denote by $a \downfree^{i}_{A} B$ if $tp^{M^{(i)}}_{\mathcal{L}}(a/AB)$ does not fork over $A$ and by $a \downfree^{ACF}_{A} B$ if $a$ is $ACF$-independent of $B$ over $A$.

**Theorem**

*Let $M$ be a PRC field with exactly $n$ orders and $T = Th_{\mathcal{L}}(M)$ with $\mathcal{L}$ the language of rings expanded by constants for a submodel.*

1. *All sets are extensions bases and forking equals dividing.*
2. *$a \downfree_{A} B$ if and only if $a \downfree^{i}_{A} B$, for all $1 \leq i \leq n$*

# Pseudo *p*-adically closed fields

The strategies used can be generalize to another class of fields, the class of pseudo p-adically closed fields (*PpC* fields). The *PpC* fields are the analogous of *PRC* fields for *p*-adic valuations, more specifically:

**Definition**

*A field M is called PpC if M is existentially closed (respect to $\mathcal{L}_{\mathcal{R}}$) into each regular field extension N to which all p-adic valuations of M extend.*

[Jarden] The class of *PpC* fields is axiomatizable in $\mathcal{L}_R$.

The *p*-adically closed fields are examples of *PpC* fields.

# Pseudo $p$-adically closed fields

**Theorem**

*Let $M$ be a bounded PpC field. Then $Th(M)$ is not NIP*

**Theorem**

*Let $M$ be a PpC with exactly $n$ $p$-adic valuation. Then $Th(M)$ is strong of burden $n$.*

# Elimination of imaginaries

**Theorem (Hrushovski)**

*Let $M$ is a bounded PAC field, and let $\mathcal{L}$ the language of rings expanded by enough constants. Then $Th_{\mathcal{L}}(M)$ eliminate imaginaries.*

**Theorem**

*Let $M$ is a bounded PRC field, and let $\mathcal{L}$ the language of rings expanded by enough constants. Then $Th_{\mathcal{L}}(M)$ eliminate imaginaries.*