From Manin-Mumford to Mordell-Lang

via "soft" model theory

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1. A simple case of Mordell-Lang
2. Some history
3. The proof and its ingredients
4. The "quantifier elimination"

Why want to find a new model-theoretic Proof of "a true result MM" ⇒ "a true result ML." when there exists already a direct Proof via Model theory of ML. (Hrushovski '93)?

Proof uses in essential way the beautiful but complicated "Dichotomy" for Zariski Geometries (Hrushovski-Zilber).

Alternative proofs (via model theory or geometry) give other information and can "shed light" on what this "dichotomy" means in this particular context...

1. Mordell-Lang

The "objects":

- $K_0 := \widetilde{\mathbb{F}_p(t)}^{sep}$ separably closed field of imperfection degree 1 (= complete theory)

K_0 not perfect

$K_0^{p^\infty} := \bigcap_n K_0^{p^n}$ biggest alg-closed subfield

here $= \widetilde{\mathbb{F}_p}^{alg}$.

any such K comes equipped with a Hasse derivation

K in the language with the Hasse derivation

or the "lambda" functions and a p-basis

has Q.E. and elimination of imaginaries

in (K, Δ) Δ a Hasse derivation

K^{p^∞} is the field of constants.

With Δ, or with the "lambda" functions

one endows K with the topology

generated by Δ-polynomials

(or λ-polynomials). Non Noetherian.

· $Th(K_0)$ is stable not superstable

K_0 is the prime model.

Abelian Varieties : K a sep. closed field of d° 1.

A is an abelian variety over :

- connected alg. group over K
- complete as a variety : $\forall Y$ variety

$Pr: A \times Y \to Y$ is closed.

$\Longrightarrow$ A $\left(= A(\widetilde{K}^{alg})\right)$ is a commutative divisible group.

- if $p \nmid n$ n-torsion $A \cong \left(\mathbb{Z}/_{n\mathbb{Z}}\right)^{2 \dim A}$
- for r, $0 \le r \le \dim A$ p^n-torsion $A \cong (\mathbb{Z}/_{p^n\mathbb{Z}})^r$

- if $r = \dim A$, A is ordinary

- $A(K_1)$ is n-divisible for $p \nmid n$.
  but <u>not</u> divisible.
- $p^\infty A(K) := \bigcap_n p^n A(K_1)$ is the biggest divisible subgroup.

  always infinite as $p^\infty A(K_1) \supseteq p'$-torsion

  Note : if $K \supseteq K_0$, $\aleph_1$-saturated then.

  $p^\infty A(K)$ contains non-torsion elements (by compactness).

  <u>What about $p^\infty A(K_0)$?</u> $\left(K_0 = \widetilde{\mathbb{F}_p(t)}^{sep}\right)$

Ex. of abelian varieties : elliptic curves, Jacobians

$\boxed{\text{not}}$: $\mathbb{G}_m$ or $\mathbb{G}_m^n$ ← Torus

semi-abelian : $0 \to \mathbb{G}_m^n \to G \to A \to 0$ ← ab. variety

. A is <u>simple</u> if A has no proper infinite closed subgroup.

ML_0: a simple case of Mordell-Lang for function fields.

Let A be a simple abelian variety over $K_0 = \widehat{\mathbb{F}_p(t)}^{sep}$
not isogenous to an ab. variety over $K_0^{p^\infty} = \widehat{\mathbb{F}_p}^{alg}$.

Let $\Gamma \subset A(K_0)$ be a subgroup of finite p'-rank:
(there is Γ_0 fin. generated, s.t. $\forall g \in \Gamma$ $ng \in \Gamma_0$ for some n prime to p).

Let $X < A$ closed irreducible.

If $\overline{X \cap \Gamma}^{Zar} = X$, then $X = A$.

Now a little history

2. Until 2011, only proof of ML_0
is Hrushovski's '93 using Zariski Geometries
(H.-Zilber) and the "dichotomy"
"one-based or a field is definable."

Existed: - Abramovich-Voloch before for ordinary '93
- Pillay-Ziegler (jet spaces) for ordinary '02

Manin-Mumford: Case when $\Gamma_0 = \{0\}$, $\Gamma = p'$-torsion

Full Manin-Mumford $\Gamma = \text{Torsion } A$

in 2005, Pink-Rössler geometric proof of Full M.M.
2006 Scanlon proof using dichotomy in $ACFA_p$.

2010: we realise that in char. 0, over $\widetilde{\mathbb{C}(t)}^{alg}$

$MM \Rightarrow ML$

and in char. $\underline{p}$: Full MM over $K_0 \Rightarrow ML_0$.

modulo

1. $p^\infty A(K_0) \subseteq$ Torsion
2. $p^\infty A(K)$ with induced structure has Q.E.

2011 D. Rössler shows (1)

and gives geometric proof of Full $MM \Rightarrow ML_0$ for Γ fin. generated

Then Corpet generalizes to all Γ

later also P. Ziegler.

= First geometric proof of ML_0.

Since: Rössler: direct proof of ML for smooth X.

Back to Model Theory:

So modulo Full MM $\quad ML_0$ reduces to ② a quantifier elimination result

Finally (2014) it was proved.

3. The proof of $MM \Rightarrow ML_0$ via Model Theory

We have Full MM $+ \; p^\infty A(K_0) \subseteq$ torsion

Want ML_0.

at first follow Hrushovski, replace Γ by an infinitely definable group.

Ideally $p^\infty A(K_0)$!

have $X < A$ closed irred. such that

$$\overline{X \cap \Gamma}^{Zar} = X$$

If: $\overline{X \cap p^{\infty} A(K_0)}^{Zar} = X$ (*)

Then. as $p^{\infty} A(K_0) \subseteq$ Torsion A

$$\overline{X \cap \mathrm{Tor}\, A}^{Zar} = X$$

M.M. applies and $X = A$.

Problem no reason (*) is true.

but if $K \geq K_0$ $\aleph_1$ - saturated

then for some coset of $p^{\infty} A(K)$, C,

$$\overline{X \cap C}^{Zar} = X .$$

Pf: as $\Gamma \leq A(K)$ has finite p'-rank

$\Gamma / p^n \Gamma$ is finite.

So $\Gamma \subseteq \bigcup_{\text{finite}}$ cosets of $p^n A(k)$

X irreducible $\Rightarrow \overline{X \cap C_n} = X$ for some coset

Then compactness and saturation

$\Rightarrow \overline{X \cap C} = X$ for some coset C of $\bigcap_n p^n A(K)$

but $\mathbb{C}(K_0)$?

<u>to simplify</u> suppose we have $\overline{X \cap p^\infty A(K)} = X$

?

$\overline{X \cap p^\infty A(K_0)} = X$

Consider $\langle p^{\infty} A(K)$, induced structure$\rangle$

: a first-order structure on $p^{\infty}A(K)$

language: for each definable set D in K

R_D a predicate interpreted as

$p^{\infty}A(K) \cap D$ (and also cartesian products)

If $\langle p^{\infty} A(K_0)$, induced str.$\rangle \preceq \langle p^{\infty}A(K)$, ind. str$\rangle$

Then $X \cap p^{\infty} A(K_0)$ dense in X

$\Longleftrightarrow$ for all $\mathcal{O}$ open in X.

$\exists y \; (y \in R_X \wedge y \in R_{\mathcal{O}})$

So suffices to show

$$\langle p^{\infty} A(K_0), \mathrm{Ind.} \rangle \preceq \langle p^{\infty} A(K), \mathrm{Ind} \rangle$$

Use :

<u>thm</u> (F. Wagner) : If G is a finite Morley rank group, g-minimal (no infinite definable proper subgroup), then if $F \subset G$ is infinite and alg. closed, $F \preceq G$.

$\boxed{\text{So } \underline{\text{need}}}$ $\langle p^{\infty} A(K), \mathrm{Ind} \rangle$ is finite Morley rank, g-minimal and $p^{\infty} A(K)$ is alg. closed in .

Know this for the quantifier free
part of $\langle p^\infty A(K), \mathrm{Ind} \rangle$

- $p^\infty A(K) \subset A(K)$ is g-minimal
  by simplicity of A (Hrushovski 93)
- $p^\infty A(K) \subset A(K)$ has "Relative Morley rank": $p^\infty A(K)$ has finite U-rank

and for all D def. $p^\infty A(K) \cap D$ contains only finitely many types of max. rank.

$\boxed{\text{But}}$ in $\langle p^\infty A(K), \mathrm{Ind} \rangle$ have
also projections (or "$\exists$" quantifiers
over these relatively definable sets).

So need a Q.E. result for $\langle p^\infty A(K), \mathrm{Ind}\rangle$.

Proposition : if K sep. closed ($\aleph_1$ - saturated) of finite degree of imperfection, A ab. variety over K, then $\langle p^\infty A(K), \mathrm{Ind}\rangle$ has Q.E.

proof uses : - finite rel. Morley rank of $p^\infty A(K)$
- Completeness of A

In fact goes through
- $p^\infty A(K)$ with the induced Λ-topology (or Δ-topology from Hasse Derivation) is noetherian of finite dimension

- Λ_completeness:

  if Z is Λ_closed in K.

  $pr: Z \times p^{\infty} A(k) \longrightarrow Z$ is Λ-closed.

Then Q.E. proceeds a lot like the proof that Zar. geometries have Q.E.

Need some assumption like $\left\{ \begin{array}{l} \text{relative Morley rank} \\ \text{completeness} \end{array} \right.$

have examples of semi-abelian varieties G where $\langle p^{\infty} G(k), \mathrm{Ind} \rangle$ does not have Q.E.

In previous paper we showed that

if $0 \to \mathbb{G}_m \to G \to E \to 0$

$\downarrow$

def. over K^{p^∞}

but G not descending to K^{p^∞}

[exists because such extensions parametrizes by $\hat{E}$ dual ab. variety.]

Then. $p^\infty G(K)$ does not have relative Morley rank : $p^\infty G(K) \cap \mathbb{G}_m(K) / p^\infty \mathbb{G}_m(K)$

has infinite (bounded) index.

$p^\infty \mathbb{G}_m(K)$ " connected component

A. Omar. Aziz studied the induced structure $\rightsquigarrow \langle p^\infty G(K), \mathrm{Ind} \rangle$ does not have Q.E.