$\widetilde{\mathfrak{M}}_c$-Groups

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We analyze the behaviors of nilpotent and almost nilpotent subgroups of $\widetilde{\mathfrak{M}}_c$-groups.

For any group, the ***Fitting subgroup*** is the group generated by all its normal nilpotent subgroups. Recall that the Fitting subgroup is not necessarily nilpotent. For groups in which every descending chain of centralizers stabilizes after finitely many steps ($\mathfrak{M}_c$-groups), nilpotency of the Fitting subgroup was shown by Bryant [3] for periodic groups, by Poizat and Wagner [12] in the stable case and in general by Derakhshan and Wagner [5]. Furthermore, it has been recently generalized by Palacín and Wagner [11] to groups type-definable in a simple theory. Such groups satisfy a weaker chain condition, namely any chain of centralizers, each having infinite index in its predecessor, stabilizes after finitely many steps. We study groups in which this chain condition holds in any definable quotient ($\widetilde{\mathfrak{M}}_c$-groups).

Making use of the definable envelops in $\widetilde{\mathfrak{M}}_c$-groups, one can show the above result in this context.

Theorem.

The Fitting subgroup of an $\widetilde{\mathfrak{M}}_c$-group is nilpotent.

Secondly, we study subgroups of $\widetilde{\mathfrak{M}}_c$-groups which are almost nilpotent. Introducing a notion of almost commutators in $\widetilde{\mathfrak{M}}_c$-groups we are able to generalize theorems of Fitting and Hall concerning nilpotent subgroups to almost nilpotent subgroups.

Theorem (Fitting's Theorem (almost nilpotent))**.**

Let H and K be two A-ind-definable almost nilpotent normal subgroups of an $\widetilde{\mathfrak{M}}_c$-group G of class n and m respectively. Then HK is an A-ind-definable almost nilpotent normal subgroup of G of class at most $n+m+1$.

Theorem (Hall's nipotency Criteria (almost nilpotent))**.**

Let N be an A-ind-definable normal subgroup of an $\widetilde{\mathfrak{M}}_c$-group G. If N is almost nilpotent of class m and $G/\widetilde{[N,N]}_A$ is almost nilpotent of class n then G is almost nilpotent of class at most $\binom{m+1}{2}n - \binom{n}{2} + 1$.

Almost centralizers

A : set of parameters
G: $|A|$-*saturated* and $|A|$-*strongly homogeneous* group
H, K, L, N denote subgroups of G.

Definition.
$H \lesssim K$ (H is *almost contained* in K) if $[H : H \cap K] < \infty$.
$H \sim K$ (*commensurate*) if $H \lesssim K$ and $K \lesssim H$.

Definition (Almost centralizer)**.**
Suppose N is normalized by H. We define:

- The *almost-centralizer* of H in K modulo N:
  $\widetilde{C}_K(H/N) = \{g \in N_K(N) : H/N \sim C_{H/N}(gN)\}$
- The *almost center* of G:
  $$\widetilde{Z}(G) = \widetilde{C}_G(G)$$
- The *nth almost center* of G inductively on n as the following:
  $$\widetilde{Z}_0(G) = \{1\} \text{ and } \widetilde{Z}_{n+1}(G) = \widetilde{C}_G(G/\widetilde{Z}_n(G))$$

Definition. A group $H = \bigcup_i H_i$ with H_i A-type-definable subgroups of G is called an A-*ind-definable* subgroup.

Symmetry and the 3-subgroup-lemma

Trivially: $H \leq C_G(K) \ \Leftrightarrow\ K \leq C_G(H)$.

In addition, for $\mathfrak{M}_c$*-groups* we have: $H \lesssim \widetilde{C}_G(K) \ \Leftrightarrow\ K \lesssim \widetilde{C}_G(H)$.

Counterexample (for arbitrary groups)**.** Let G be a finite non-commutative group, $K = \prod_\omega G$ and $H = \bigoplus_\omega G$. Then we have that $H \leq \widetilde{C}_G(G)$ but $G \not\lesssim \widetilde{C}_G(H)$.

Proposition (Symmetry for ind-definable groups)**.**
Let H and K be two A-ind-definable subgroups of G and N a subgroup of G which is the union of definable sets and normalized by H and K. Then

$$H \lesssim \widetilde{C}_G(K/N) \quad \text{if and only if} \quad K \lesssim \widetilde{C}_G(H/N).$$

The three-subgroup-lemma states:

$$[H, K, L] \leq M \text{ and } [K, L, H] \leq M \Rightarrow [L, H, K] \leq M,$$

or equivalently for M trivial:

$$H \leq C_G(K/C_G(L)) \text{ and } K \leq C_G(L/C_G(H)) \Rightarrow L \leq C_G(H/C_G(K)).$$

Proposition (Almost Three-Subgroup-Lemma)**.**
Let H, K and L be three A-ind-definable normal subgroups of G, then

$$H \lesssim \widetilde{C}_G(K/\widetilde{C}_G(L)) \text{ and } K \lesssim \widetilde{C}_G(L/\widetilde{C}_G(H))$$
$$\Rightarrow L \lesssim \widetilde{C}_G(H/\widetilde{C}_G(K)).$$

Almost commutator

Let H and K be two A-ind-definable normal subgroups of G.

Definition (Almost A-commutator)**.**

$$\tilde{[}H, K\tilde{]}_A := \bigcap\{L \trianglelefteq H : L \text{ is } A\text{-def. } H \lesssim \widetilde{C}_G(K/L)\}$$

For H, K, N and L A-ind-definable normal subgroups of G, we have the following properties:

- $\tilde{[}H, K\tilde{]}_A = \tilde{[}K, H\tilde{]}_A$
- $H \lesssim \widetilde{C}_G\left(K/\tilde{[}H, K\tilde{]}_A\right)$
- $\tilde{[}HK, N\tilde{]}_A \leq \tilde{[}H, N\tilde{]}_A \cdot \tilde{[}K, N\tilde{]}_A$
- If $N \lesssim H$ and $L \lesssim K$ then $\tilde{[}N, L\tilde{]}_A \leq \tilde{[}H, K\tilde{]}_A$
- $\tilde{[}H, K\tilde{]}_A \trianglelefteq G$

$\widetilde{\mathfrak{M}}_c$ - groups

Definition.

A group G is an $\widetilde{\mathfrak{M}}_c$*-group* if for any definable normal subgroup H there exists natural numbers n_H and d_H such that any sequence of centralizers

$$C_{G/H}(g_0H) \geq \ldots \geq C_{G/H}(g_0H, \ldots, g_mH) \geq \ldots$$

each having index at least d_H in its predecessor has length at most n_H.

The crucial property of $\widetilde{\mathfrak{M}}_c$-groups is the following:

Lemma. *Let H be an A-invariant subgroup of an $\widetilde{\mathfrak{M}}_c$-group G. Then all iterated almost centralizers $\widetilde{C}^n_G(H)$ are definable.*

This implies that

$$H \lesssim \widetilde{C}_G(K/\widetilde{C}_G(L)) \text{ if and only if } \tilde{[}H, K, L\tilde{]} = \{1\},$$

which yields:

Corollary.

Let H, K and L be three A-ind-definable normal subgroups of G. Then for any intersection M of A-definable normal subgroups of G, we have that

$$\tilde{[}H, K, L\tilde{]} \leq M \text{ and } \tilde{[}K, L, H\tilde{]} \leq M$$
$$\Rightarrow \tilde{[}L, H, K\tilde{]} \leq M.$$

Almost nilpotent subgroups

Definition (Almost nilpotent subgroup)**.**

A subgroup H of G is *almost nilpotent* if there is an *almost central series* of finite length, i. e. a sequence of normal subgroups of H

$$\{1\} \leq H_0 \leq H_1 \leq \cdots \leq H_n = H$$

s. t. H_{i+1} is a subgroup of $\widetilde{C}_H(H/H_i)$ $\forall i < n$. The least such $n \in \omega$ is the *almost nilpotency class* of H.

DEFINABLE ENVELOPS

Lemma.

Let H be a soluble subgroup of an $\widetilde{\mathfrak{M}}_c$-group G. Then there is a definable solvable subgroup S of G containing H, and a series of definable S-invariant subgroups

$$\{1\} \leq S_1 \leq \ldots \leq S_n \leq S$$

all normalized by $N_G(H)$ s. t. S_i/S_{i-1} is abelian $\forall i \leq n$.

Lemma.

Let H be an almost nilpotent subgroup of an $\widetilde{\mathfrak{M}}_c$-group G. Then there exists a definable nilpotent subgroup N of G such that $H \lesssim N$, and a series of definable N-invariant sugroups

$$\{1\} \leq N_1 \leq \ldots \leq N_n \leq S$$

all normalized by $N_G(H)$, s. t. $N_{i+1} \leq C_G(N/N_i)$ $\forall i \leq n$.

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