Singularity of the adjacency matrix of a random digraph

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Introduction and Notations

- G a graph on n vertices (loops allowed). Its adjacency matrix $A = (a_{i,j})$ is defined by

$$a_{i,j} = 1 \text{ if } i \to j \quad \text{and} \quad 0 \quad \text{otherwise.}$$

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- d-regular directed graph on n vertices $\Leftrightarrow$ each vertex has exactly d in-neighbors and d out-neighbors.

  Adjacency matrix: "d" double stochastic matrix.

  $\mathcal{D}_{n,d}$ = set of all directed d-regular graphs on n vertices

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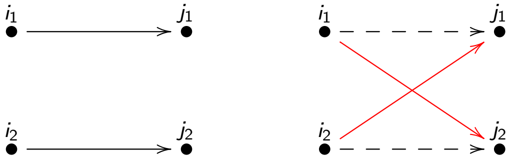
- **McKay**'81 used switching to estimate cardinalities.



Conjecture

$B = (\xi_{i,j})_{i,j}$ *iid Bernoulli* ± 1. $\mathbb{P}\{B \text{ is singular }\} = \left(\frac{1}{2} + o(1)\right)^n$.

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Komlós'77: $O(n^{-\frac{1}{2}})$; **Kahn-Komlós-Szemerédi**'95: 0.999^n;
Tao-Vu'06: $\left(3/4 + o(1)\right)^n$; **Bourgain-Vu-Wood**'09: $\left(\frac{1}{\sqrt{2}} + o(1)\right)^n$.

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Conjecture (**Vu**'08)

$\widetilde{M}$: $n \times n$ adjacency of uniform d-regular graph. For any $3 \le d \le n/2$, $\quad \mathbb{P}\{\widetilde{M} \text{ is singular }\} \underset{n\to\infty}{\longrightarrow} 0$.

Also mentioned in **Vu**'s 2014 ICM talk, **Frieze**'s 2014 ICM talk.

M: $n \times n$ adjacency of uniform d-regular directed graph.

Theorem (Cook'14)

There exists an absolute constant $c \leq 1/20$ such that for $\ln^2 n \ll d \leq n/2$,

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Theorem (**LLTTY**'15)

There exist absolute constants c, C such that for $C \leq d \leq cn/\ln^2 n$,

$$\mathbb{P}\{M \textit{ is singular }\} \leq C\frac{\ln^3 d}{\sqrt{d}}.$$

Decrypting the proof

Key ingredient: **Littlewood-Offord** anti-concentration.

Theorem (**Erdös**'43)

$\xi = (\xi)_{i \le n}$ *iid Bernoulli* ± 1*. Then for any* $a \in \mathbb{R}^n$,

$$\mathbb{P}\{\langle \xi, a\rangle = 0\} = O\left(1/\sqrt{|supp a|}\right)$$

Kleitman, Halasz, Stanley, Sarkozy-Szemeredi, Frankl-Furedi, Tao-Vu, Rudelson-Vershynin, Friedland-Sodin, Nguyen-Vu, Friedland-Giladi-Guédon, Costello, Meka-Nguyen-Vu.

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In our setting (uniform model): Fixing $R_2, \ldots, R_n$ completely determines R_1.

Cook's idea: Fix $R_3, \ldots, R_n$. The support of $R_1 \cup R_2$ is completely determined. But there is randomness left in the choice of R_1, R_2.

For $(i,j) \in [n]^2$, $\quad V_{i,j} := \mathrm{span}\{R_k\}_{k \neq i,j}$.

$$\mathrm{rk}\, M = n-1 \Rightarrow \exists (i,j) \quad \dim V_{i,j} = n-2 \ \text{ and } \ R_i + R_j \notin V_{i,j}$$

$$\Rightarrow \exists (i,j) \quad \mathrm{Ker}\, M = \{V_{i,j}, R_i + R_j\}^{\perp} = \{v_{i,j}\}$$

$$\Rightarrow \exists (i,j) \quad v_{i,j} \in \mathrm{Ker}\, M \quad (\Leftrightarrow \langle v_{i,j}, R_i \rangle = 0).$$

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$v_{i,j}$ is completely determined by $V_{i,j}$.

Conditioned on $V_{i,j}$, it is stable under switching in R_i, R_j.

Goal: Conditioned on $V_{i,j}$, with small probability $\mathrm{Ker}\, M$ is stable under switching in R_i, R_j.

Problems

Problem 1: Avoid the union bound. Otherwise we would get

$$\mathbb{P}\{\operatorname{rk} M = n-1\} \leq n^2 \mathbb{P}\{v_{1,2} \in \operatorname{Ker} M\}$$

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Problem 3: Need $v_{1,2}$ not mainly one valued on $\operatorname{supp} R_1 \cup \operatorname{supp} R_2$.

Otherwise $\langle v_{1,2}, R_1\rangle$ becomes invariant under switching.

Suppose Problems 1, 2, 3 settled and condition on $V_{1,2}$.

- Prob 1$\Rightarrow$ only care for $\mathbb{P}\big\{\langle v_{1,2}, R_1\rangle = 0\big\}$.
- Prob 2$\Rightarrow$ R_1, R_2 are (almost) disjoint.
- Prob 3$\Rightarrow$ $v_{1,2}$ is two valued on $I := \operatorname{supp} R_1 \cup \operatorname{supp} R_2$.

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Write $I = I_1 \sqcup I_2, \quad |I| = 2d, \quad |I_1| = |I_2| = d$.

Almost all coordinates of $v_{1,2}$ in $I_1 \neq$ coordinates of $v_{1,2}$ in I_2.

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Goal: $\mathbb{P}\{\langle v_{1,2}, R_1\rangle = 0\}$ is small.

$\longrightarrow$ Reconstruct R_1 and check $\langle v_{1,2}, R_1\rangle$.

Reconstruct R_1: flip d coins ε_i independently.

If $\varepsilon_i = 1$: Put 1 on the *ith* component of R_1 in I_1.

If $\varepsilon_i = 0$: Put 1 on the *ith* component of R_1 in I_2.

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We formed (almost) d independent r.v

$$\xi_i = \begin{cases} \text{ith coordinate of } v_{1,2} \text{ in } I_1 \text{ with prob } 1/2 \\ \\ \text{ith coordinate of } v_{1,2} \text{ in } I_2 \text{ with prob } 1/2 \end{cases}$$

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Erdös anti-concentration$\Rightarrow$

$$\mathbb{P}\{\langle v_{1,2}, R_1\rangle = 0\} = \mathbb{P}\{\sum_i \xi_i = 0\} = O(1/\sqrt{\sim d}) = O(\ln^3 d/\sqrt{d}).$$

The proof is done!

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In particular, if $J = \{i, j\}$, we needed $|\mathcal{N}_G^{out}(J)|$ to be big.

Theorem (LLTTY'15)

Let $8 \le d \le n/12$, $\varepsilon \in (0,1)$ *and* $k \le c\varepsilon n/d$. *Then with probability* $1 - \exp\left(-c\varepsilon^2 dk \ln\left(e\varepsilon n/dk\right)\right)$ *we have*

$$\forall J \subset [n] \text{ of size } |J| = k, \quad \left| |\mathcal{N}_G^{out}(J)| - d|J| \right| \le \varepsilon d|J|.$$

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Remark

- *If $i, j \in [n]$ are two vertices. With high probability,*

$$|\mathcal{N}_G^{out}\{i,j\}| \ge 2(1-\varepsilon)d.$$

$\Rightarrow$ *at most $2\varepsilon d$ common out-neighbors to i and j.*

$\Rightarrow$ *R_i, R_j almost disjoint (Prob 2 settled).*

- **Cook**'*14: Concentration inequalities for codegrees when $d \gg \log n$.*

Connectivity of large sets

Theorem (LLTTY'15)

There exist absolute positive constants c, C such that the following holds. Let $C \leq d \leq cn$ and let natural numbers ℓ and r satisfy

$$\frac{n}{4} \geq r \geq \ell \geq \frac{Cn\ln(en/r)}{d}.$$

Then

$$\mathbb{P}\left\{\bigcup\{I \not\to J\}\right\} \leq \exp\left(-\frac{cr\ell d}{n}\right),$$

where the union is taken over all $I, J \subset [n]$ with $|I| \geq \ell$ and $|J| \geq r$.

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- Theorem 5$\Rightarrow$

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- Undirected setting:**Bollobàs**'81,**Mckay**'87,**Frieze-Luczak**'92,**Krivelevich-Sudakov-Vu-Wormald**'01,**Cooper-Frieze-Reed-Riordan**'02.

No almost constant null vectors

For $0 < p < 1/2$ consider the following set of vectors

$$AC(p) = \{x \in \mathbb{R}^n \setminus \{0\} \ : \ \exists \lambda_x \in \mathbb{R} \quad |\{i \ : \ x_i = \lambda_x\}| \geq (1-p)n\}.$$

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Theorem (LLTTY'15)

There are absolute positive constants C, c such that for $C \leq d \leq cn$ and $p \leq c/\ln d$ one has

$$\mathbb{P}\left\{\mathrm{Ker} M \cap AC(p) = \emptyset\right\} \geq 1 - \left(\frac{Cd}{n}\right)^{cd}.$$

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$x \notin AC(p) \Rightarrow \exists J,\ pn \leq |J| \leq (1-p)n, \quad \forall i \in J,\ j \in J^c \ \ x_i \neq x_j.$

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$x \notin AC(p) \Rightarrow \exists J$, $pn \leq |J| \leq (1-p)n$, $\quad \forall i \in J,\ j \in J^c \quad x_i \neq x_j$.

$(q \sim d/\ln^2 d)$. No $p/2q \times p/2$ zero minor $\Rightarrow$

$$|\{i \ : \ |\operatorname{supp} R_i \cap J| \geq q\}| \geq (1 - p/2q)n$$

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Conclusion: x is two valued on the support of almost all rows.

$y \notin AC(p) \Rightarrow \exists$ at least pn indices s.t $y_i \neq 0$.

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$$v_{i,j} \text{ not one valued on } \mathrm{supp}\, R_i \cup \mathrm{supp}\, R_j.$$

Prob 1, 3 settled!

Ruling out AC null vectors

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- Easy to rull out very sparse vectors: If $m_0 \ll n/d$ and $x_{m_0} = 0$.

  Expansion $\Rightarrow \exists$ many rows R_i ($\sim m_0 d/4$) with one 1 on $[m_0]$.

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- Now $x_{m_0} \neq 0$, rescale so $|x_{m_0}| = 1$ and suppose $\lambda \geq 0$.

$$J_\lambda := \{i \mid x_i = \lambda\}, \quad J := \{i \geq m_0 \mid |x_i - \lambda| > 1/2d\}$$

We have $|J_\lambda| \sim (1-p)n$, $|J| \sim ?$

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  - $\lambda \leq 1/2d \Rightarrow$ we have a big jump at $m_1 := m_0 + |J|$.

    $|x_{m_0}| = 1$ while $|x_{m_1}| \leq 1/d$.

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Expansion$\Rightarrow \exists$ row R_i with one 1 on $[m_0]$ and 0 on J.

$$\Rightarrow |(Mx)_i| \geq |x_{m_0}| - (d-1)|x_{m_1}| > 0.$$

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- If $|J| \ll n/d$, then it is easy.
  - $\lambda > 1/2d \Rightarrow$ few negative coord in x, easy to avoid them.
  - $\lambda \leq 1/2d \Rightarrow$ we have a big jump at $m_1 := m_0 + |J|$.

$$|x_{m_0}| = 1 \text{ while } |x_{m_1}| \leq 1/d.$$

Expansion$\Rightarrow \exists$ row R_i with one 1 on $[m_0]$ and 0 on J.

$$\Rightarrow |(Mx)_i| \geq |x_{m_0}| - (d-1)|x_{m_1}| > 0.$$

- $|J| \sim n/d$, we have a partition $I \sqcup J \sqcup J_\lambda = [n]$ s.t

$$x_{m_0} = 1, \quad \forall \ell \in J_\lambda \;\; x_\ell = \lambda \quad \text{and} \quad \forall j \in J \;\; x_j - \lambda \geq 1/2d. \tag{1}$$

Show that $\mathbb{P}\{\exists x \text{ satisfying (1) s.t. } \|Mx\|_\infty = 0\} \sim 0$.

Net argument: outside $[m_0]$, special net on the cube.

Fix y satisfying (1), $S_0 =$ rows which are 0 on $[m_0]$.

Goal: $\mathbb{P}\left\{ \|P_{S_0} M y\|_\infty < 1/4d \mid I \text{ fixed } \right\} \leq \exp(-n)$.

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How does it translate in terms of connectivity of the graph?

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Fix I and $i \in S_0$,

$$\{M, |(My)_i| < 1/4d\} \subseteq \{M, i \to J\}$$

or

$$\{M, |(My)_i| < 1/4d\} \subseteq \{M, i \not\to J\}$$

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Conclusion: Matrices satisfying $\|P_{S_0} My\|_\infty < 1/4d$ share the same configuration of edges from S_0 to J.

Anti-concentration

Define

$$\delta_i^J = 1 \text{ if } i \to J \quad \text{and} \quad 0 \text{ otherwise.}$$

Set $\delta^J = (\delta_1^J, \ldots, \delta_n^J) \in \{0,1\}^n$.

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Theorem 3 (LLTTY'15)

Let $8 \leq d \leq cn$ and $I, J \subset [n]$ disjoint such that

$$|I| \leq \frac{d|J|}{8} \quad \textit{and} \quad |J| \leq c\frac{n}{d}.$$

Let $F \subset [n] \times [n]$. For any $v \in \{0,1\}^n$,

$$\mathbb{P}\{\delta^J = v \mid E_G^{in}(I) = F\} \leq \exp\left(-cd|J| \ln \frac{n}{d|J|}\right).$$

What's next?

- Show the conjecture for fixed d.
- Limiting spectral distribution.

  Conjecture (**Chafai-Bordenave**): The limiting spectral empirical measure is given by

  $$\frac{1}{\pi}\frac{d^2(d-1)}{(d^2-|z|^2)^2}\mathbf{1}_{\{|z|<\sqrt{d}\}}dxdy$$

  This is the non-symmetric version of Kesten-Mckay measure.

- The non-directed version of our main theorem.
  $\to$ In progress with **LLTT**.

Thank you

Happy New Year!