Spectra of large diluted but bushy random graphs

Laurent Ménard
Joint work with Nathanaël Enriquez
Modal'X Université Paris Ouest

Erdős-Rényi random graphs

$G(n, p)$

- vertex set $\{1, \ldots n\}$
- vertices linked by an edge independently with probability p

Erdős-Rényi random graphs

$G(n, p)$

- vertex set $\{1, \ldots n\}$
- vertices linked by an edge independently with probability p

Adjacency matrix A

- symmetric
- if $i \neq j$, $P(A_{i,j} = 1) = 1 - P(A_{i,j} = 0) = p$
- for every i, $A_{i,i} = 0$

Erdős-Rényi random graphs

$G(n, p)$

- vertex set $\{1, \dots n\}$
- vertices linked by an edge independently with probability p

Adjacency matrix A

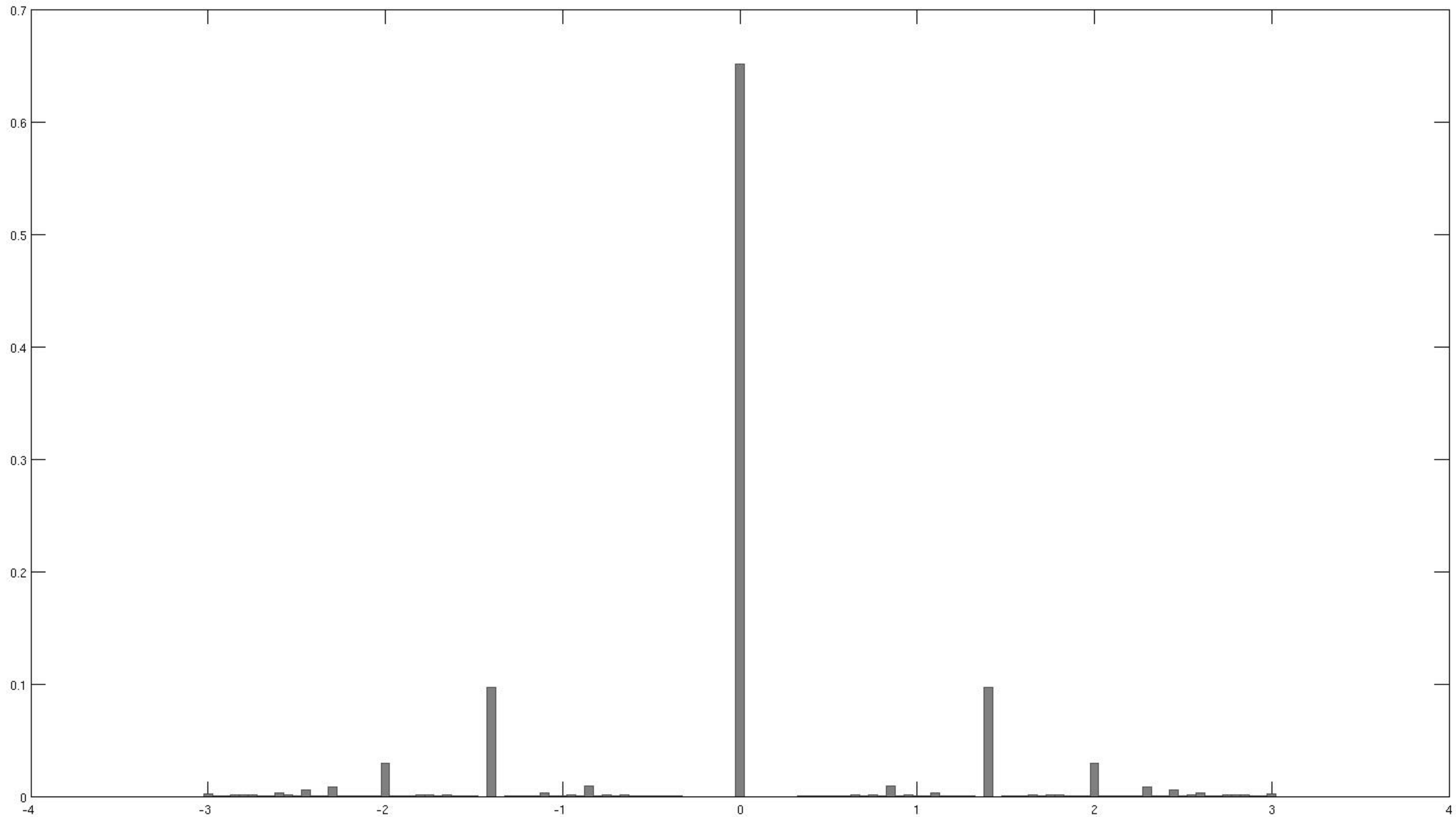
- symmetric
- if $i \neq j$, $P(A_{i,j} = 1) = 1 - P(A_{i,j} = 0) = p$
- for every i, $A_{i,i} = 0$

What does the spectrum of A look like ?

- if $np \to 0$, single atom mass at 0
- if $np \to \infty$, semi circle law
- if $np \to c > 0$, not much is known...

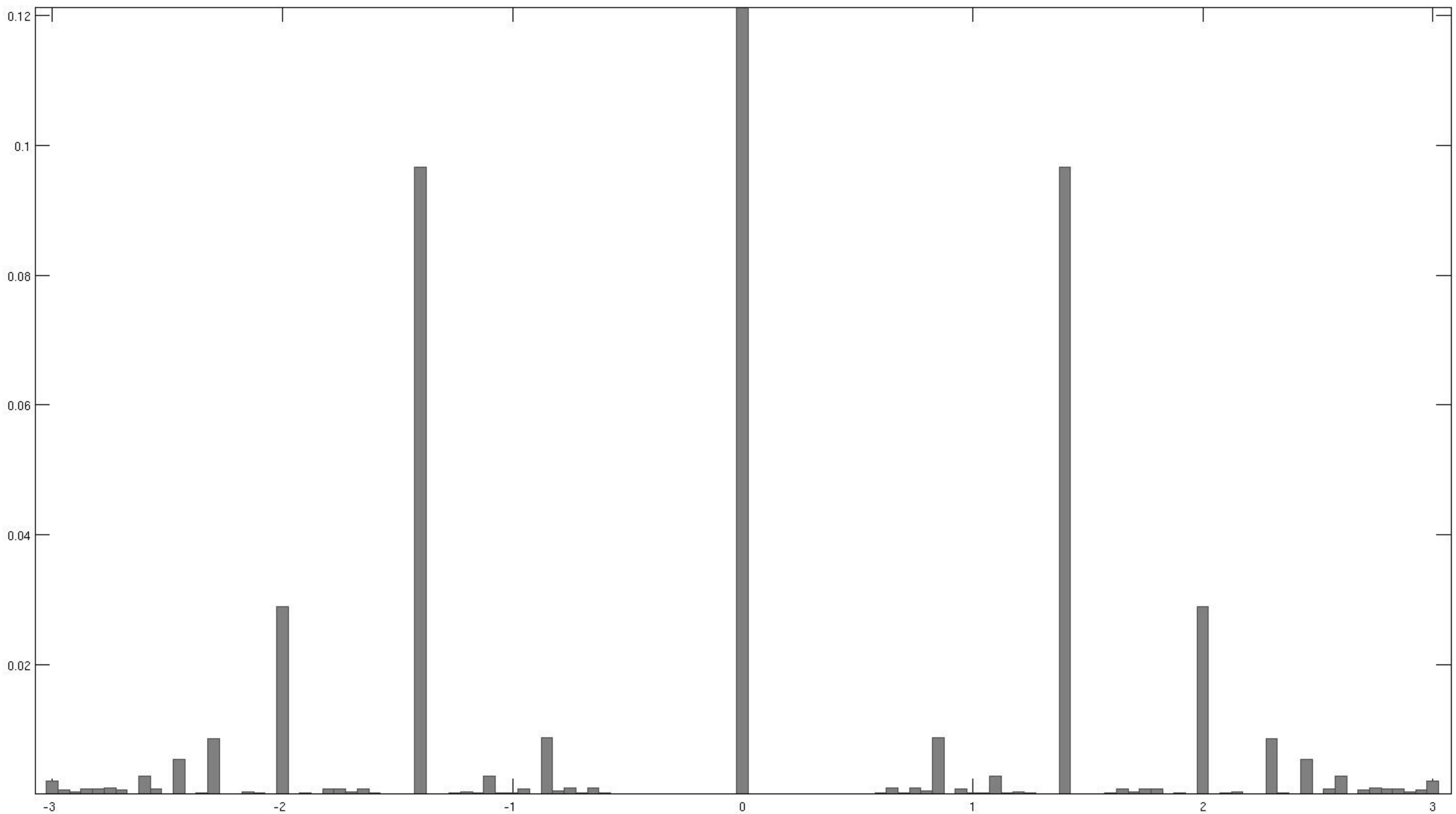
Numerical simulations on diluted graphs with 5000 vertices

$$c = 0,5$$



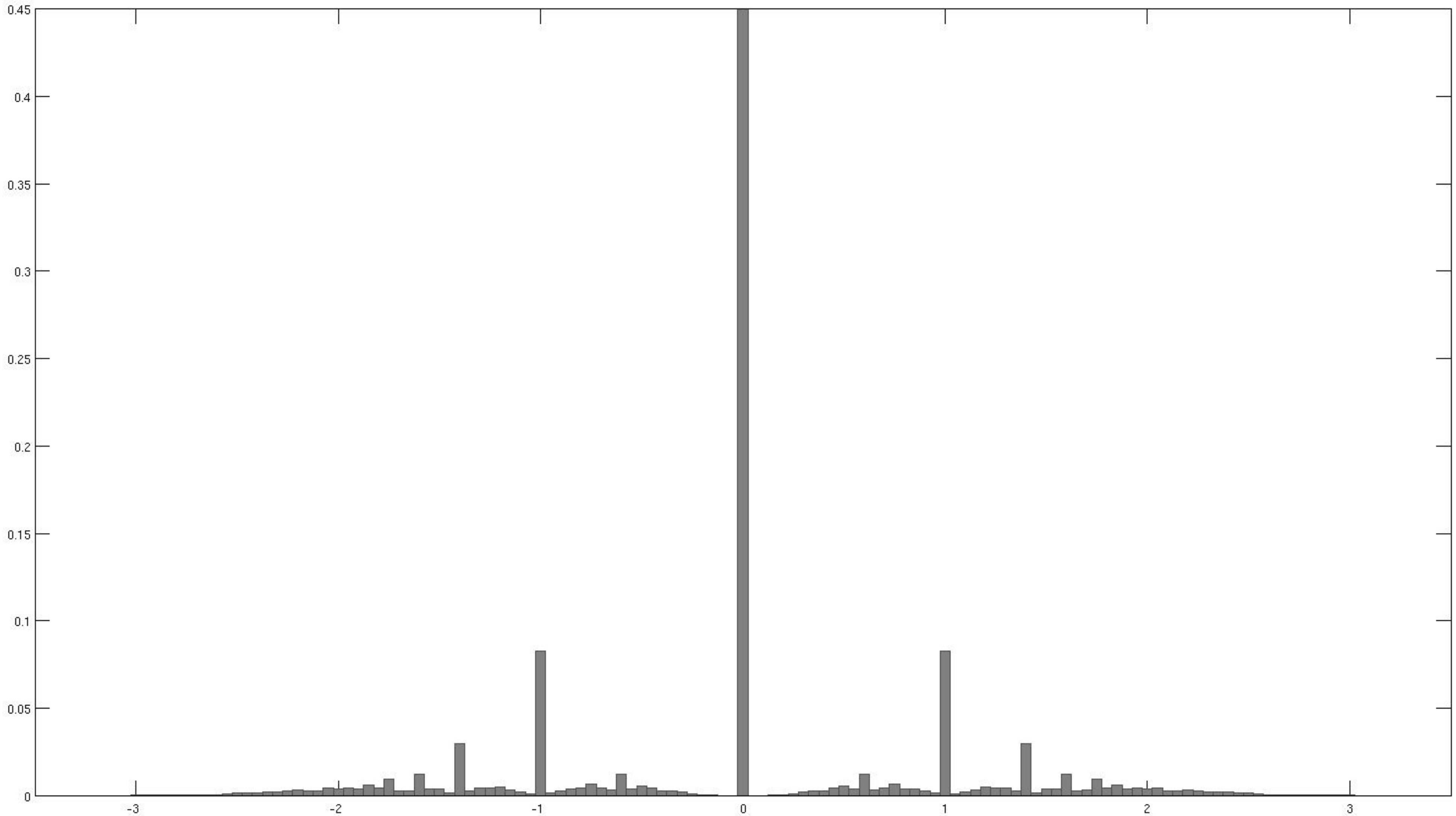
Numerical simulations on diluted graphs with 5000 vertices

$c = 0,5$ (zoomed in)



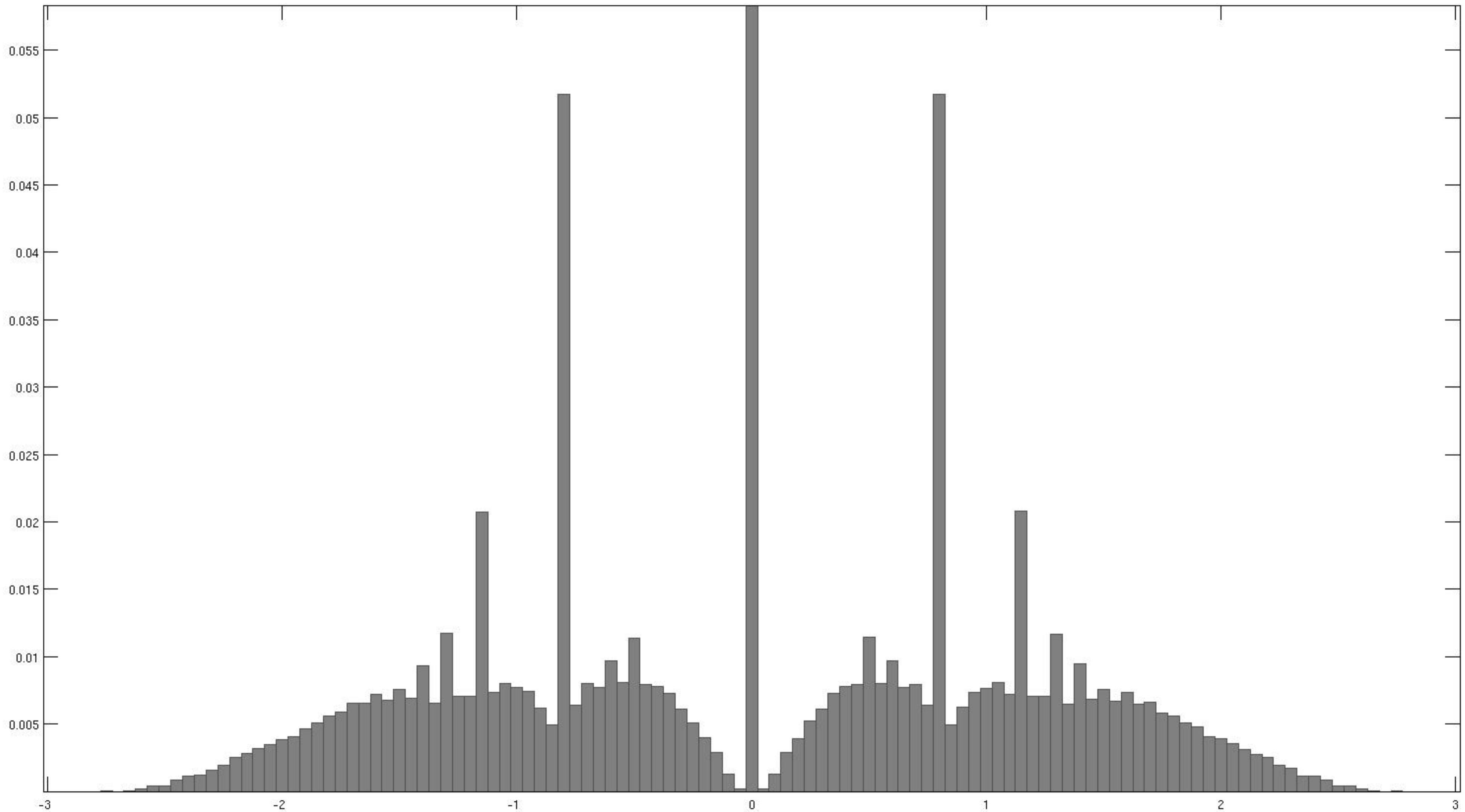
Numerical simulations on diluted graphs with 5000 vertices

$$c = 1$$



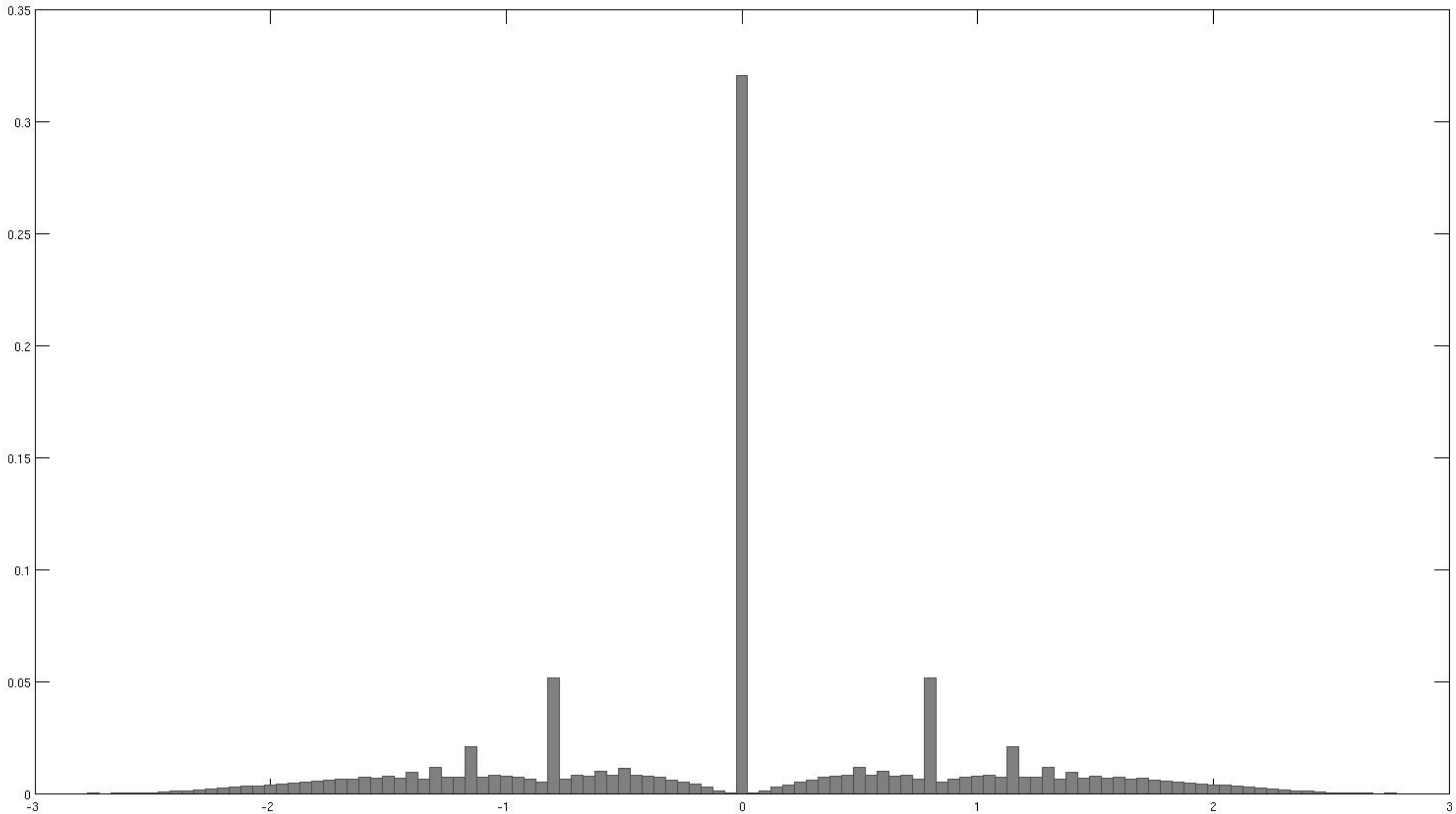
Numerical simulations on diluted graphs with 5000 vertices

$c = 1$ (zoomed in)



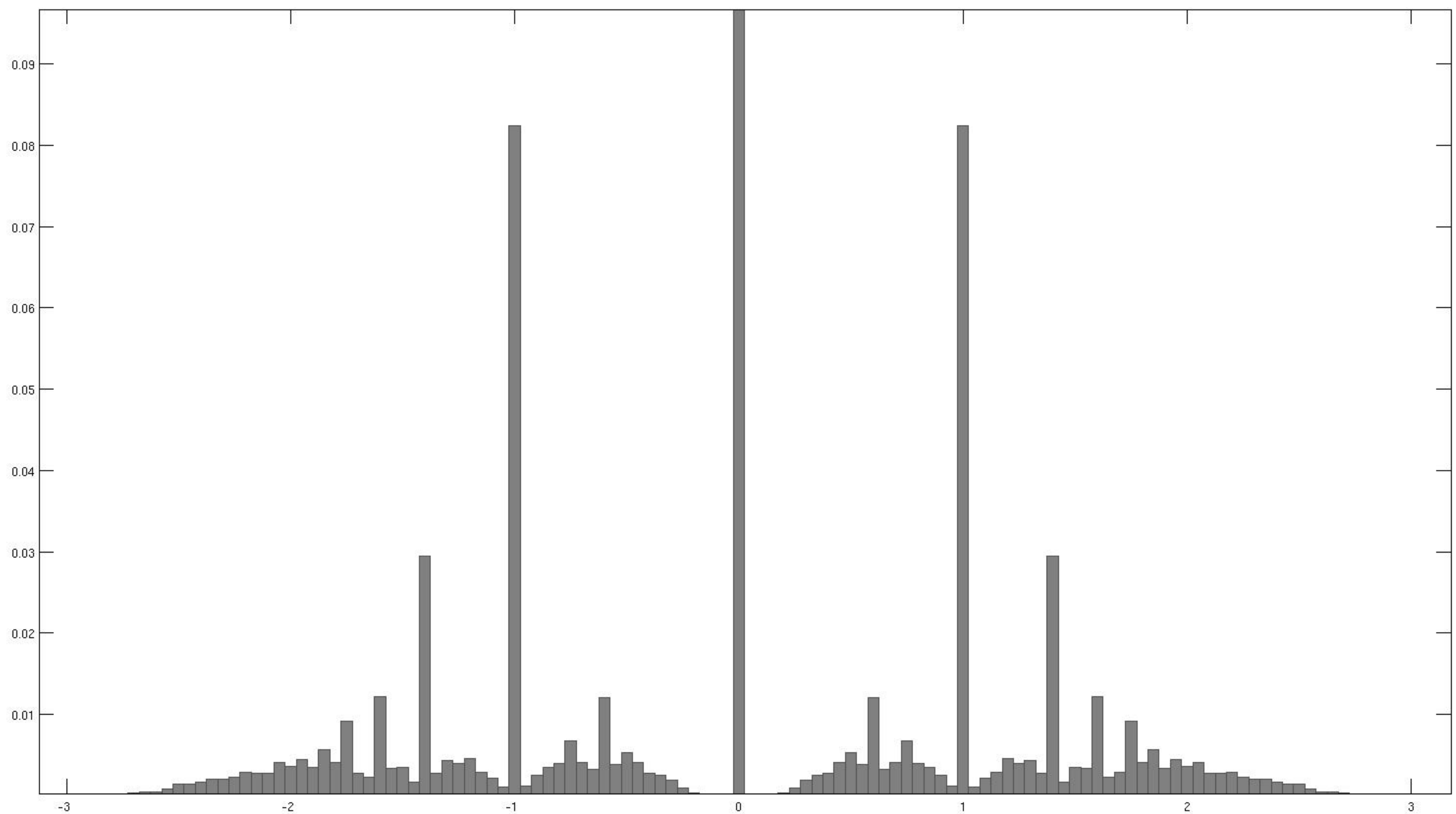
Numerical simulations on diluted graphs with 5000 vertices

$$c = 1,5$$



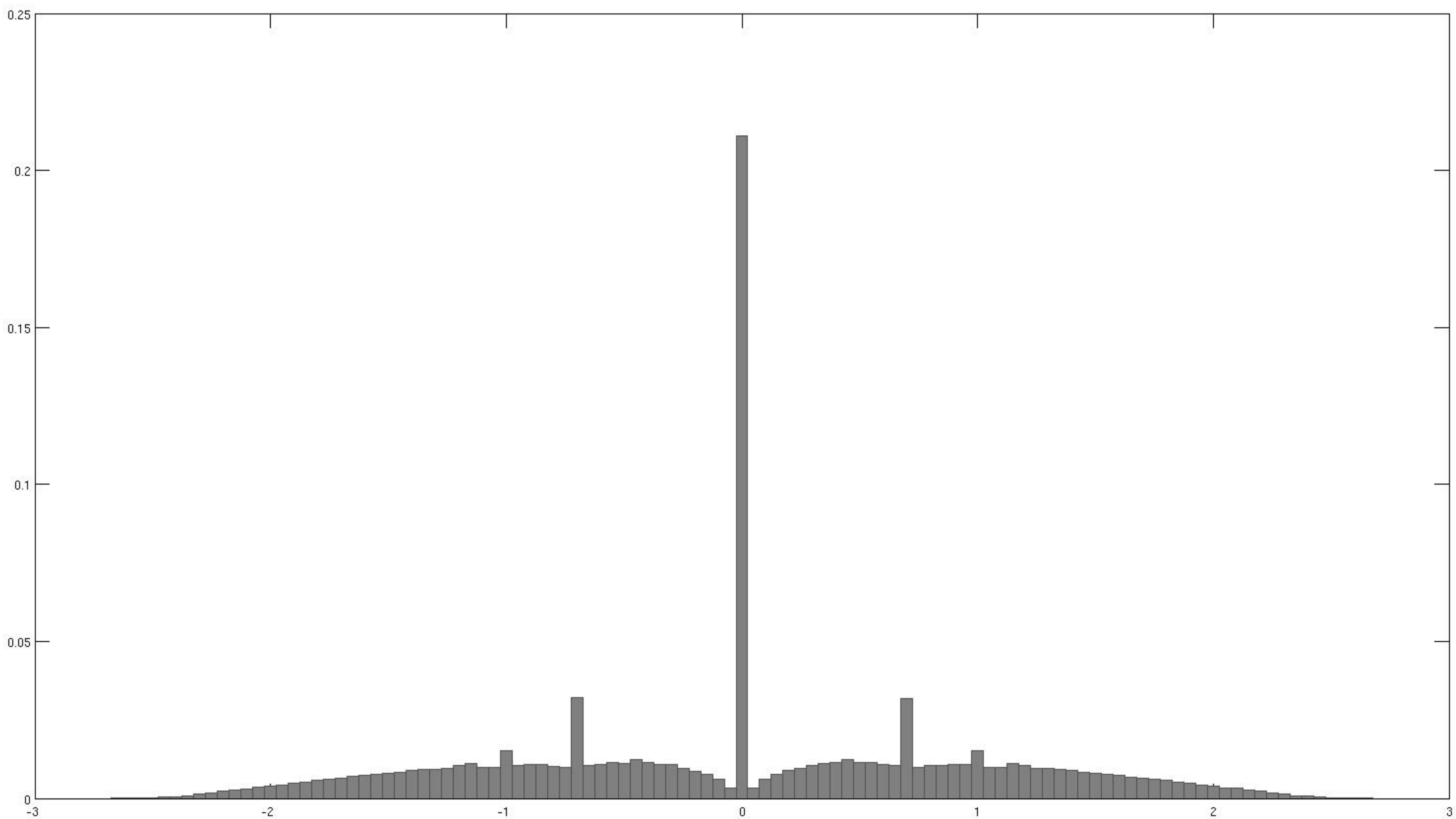
Numerical simulations on diluted graphs with 5000 vertices

$c = 1,5$ (zoomed in)



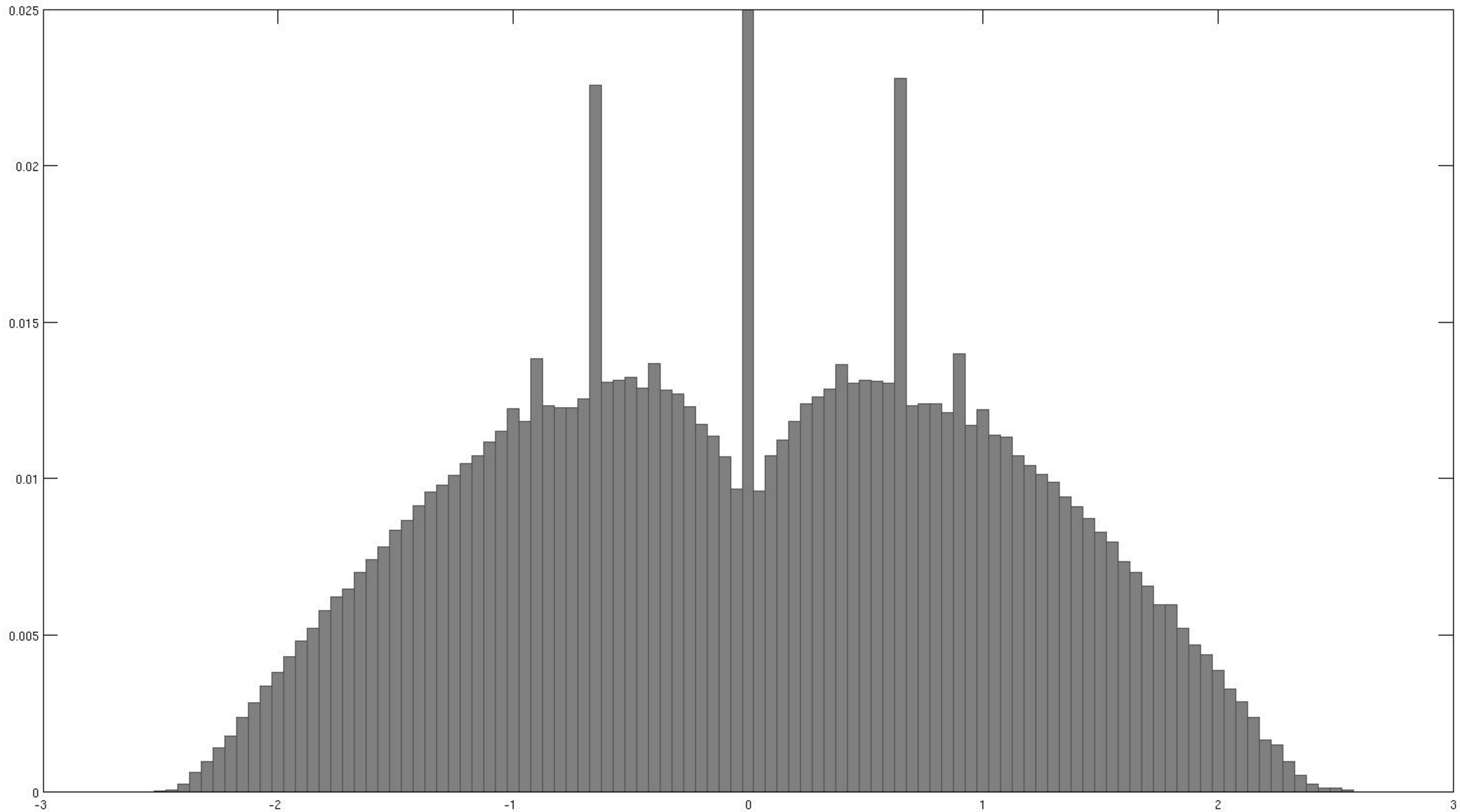
Numerical simulations on diluted graphs with 5000 vertices

$$c = 2$$



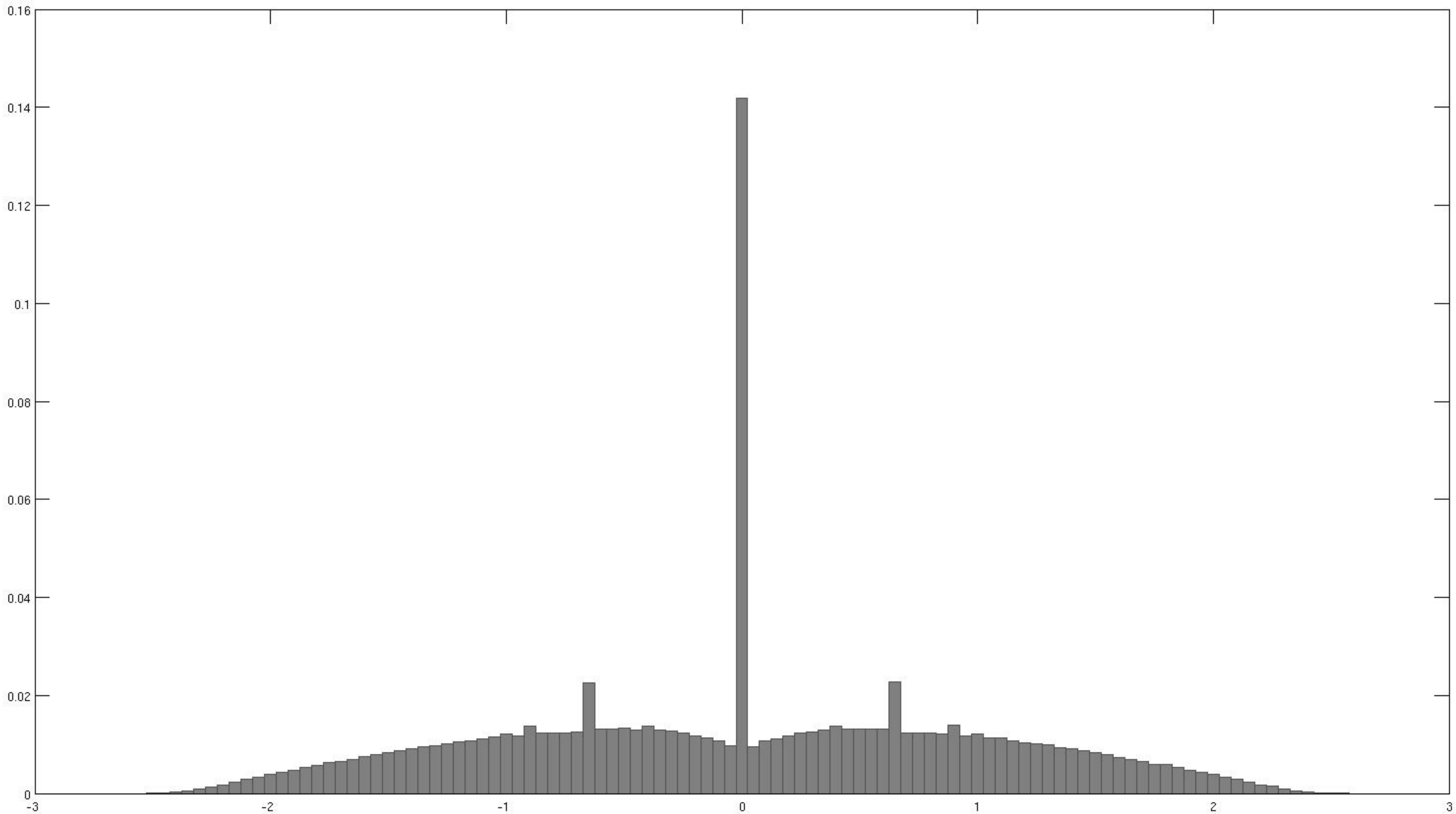
Numerical simulations on diluted graphs with 5000 vertices

$c = 2$ (zoomed in)



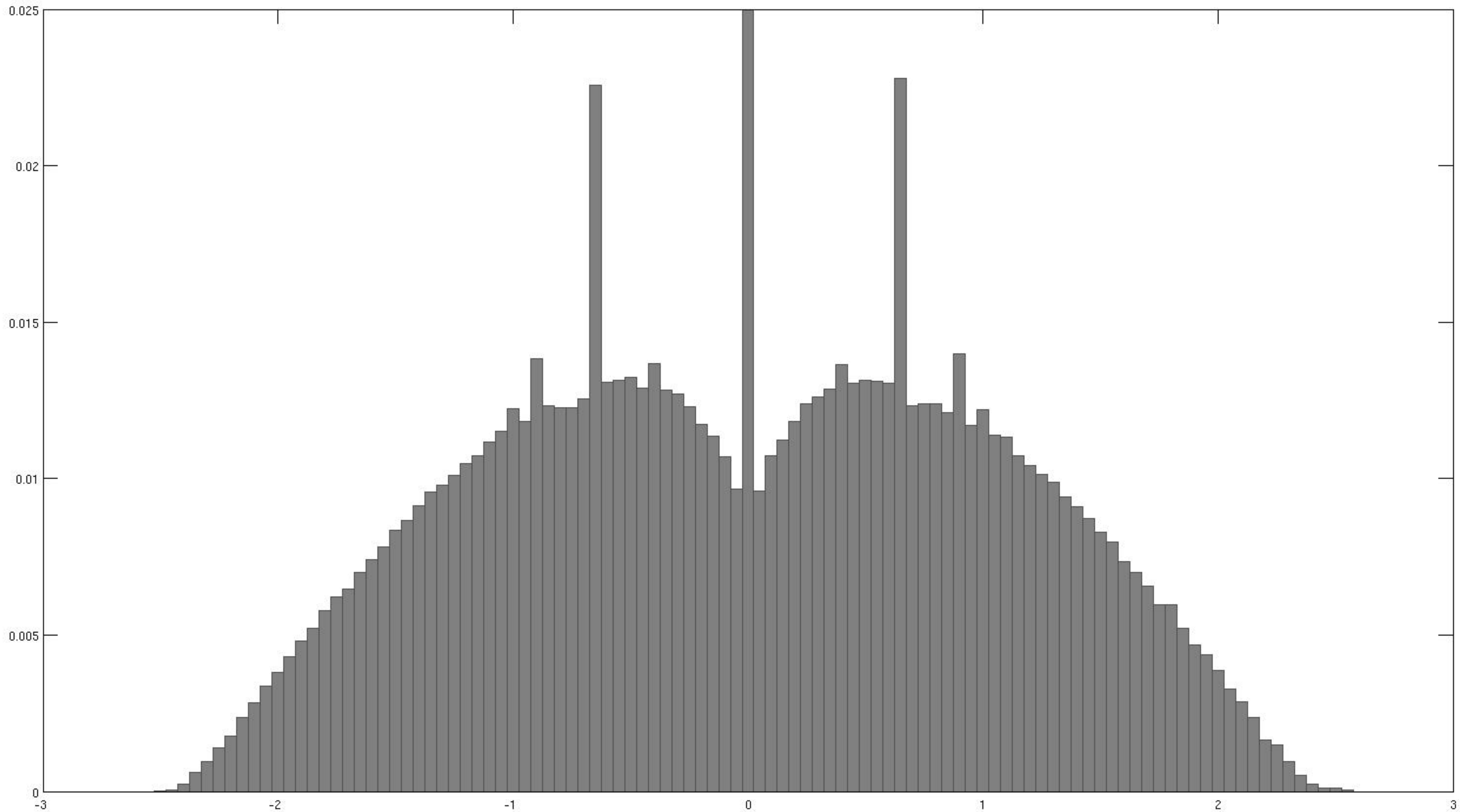
Numerical simulations on diluted graphs with 5000 vertices

$$c = 2,5$$



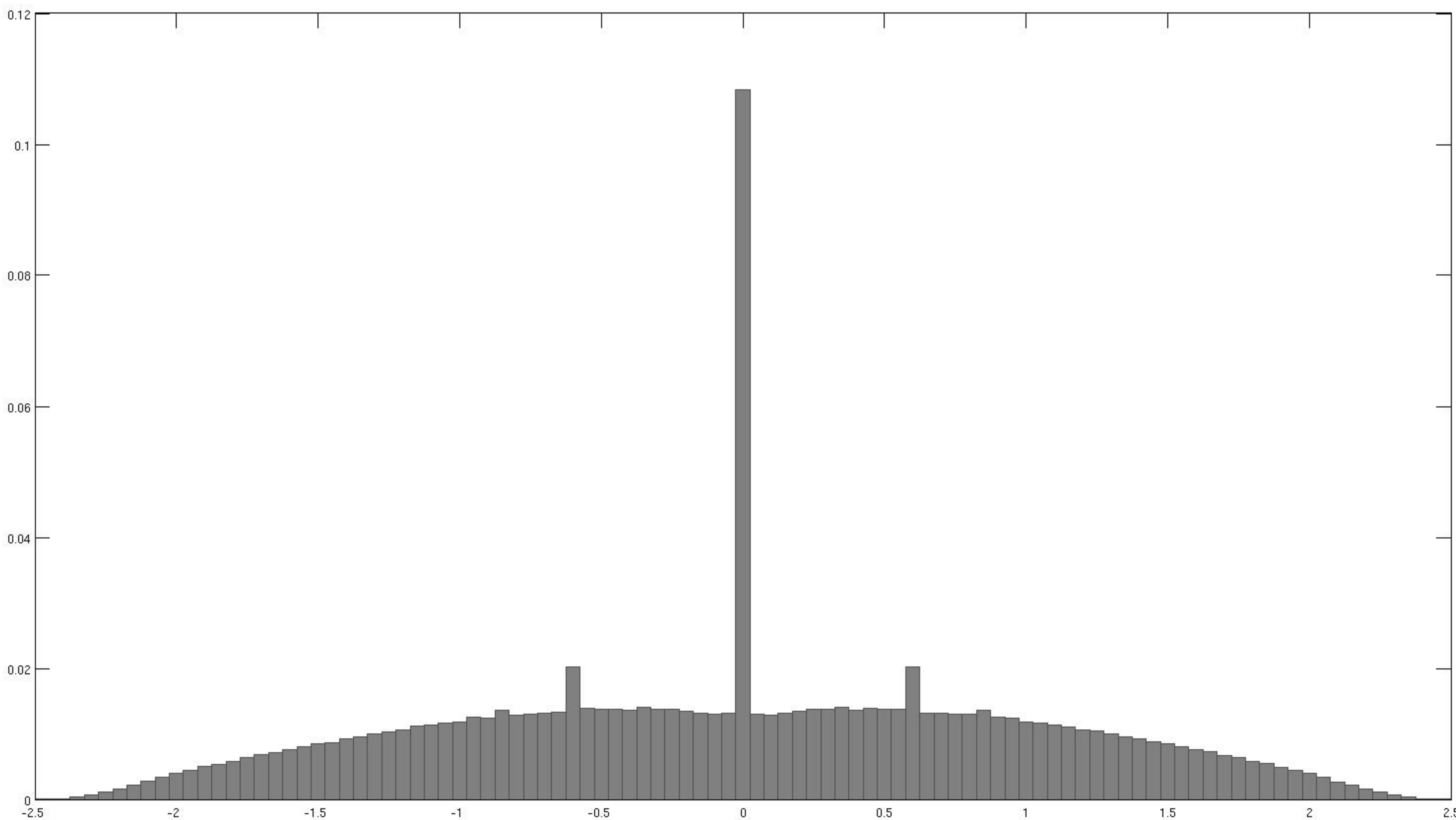
Numerical simulations on diluted graphs with 5000 vertices

$$c = 2,5 \text{ (zoomed in)}$$



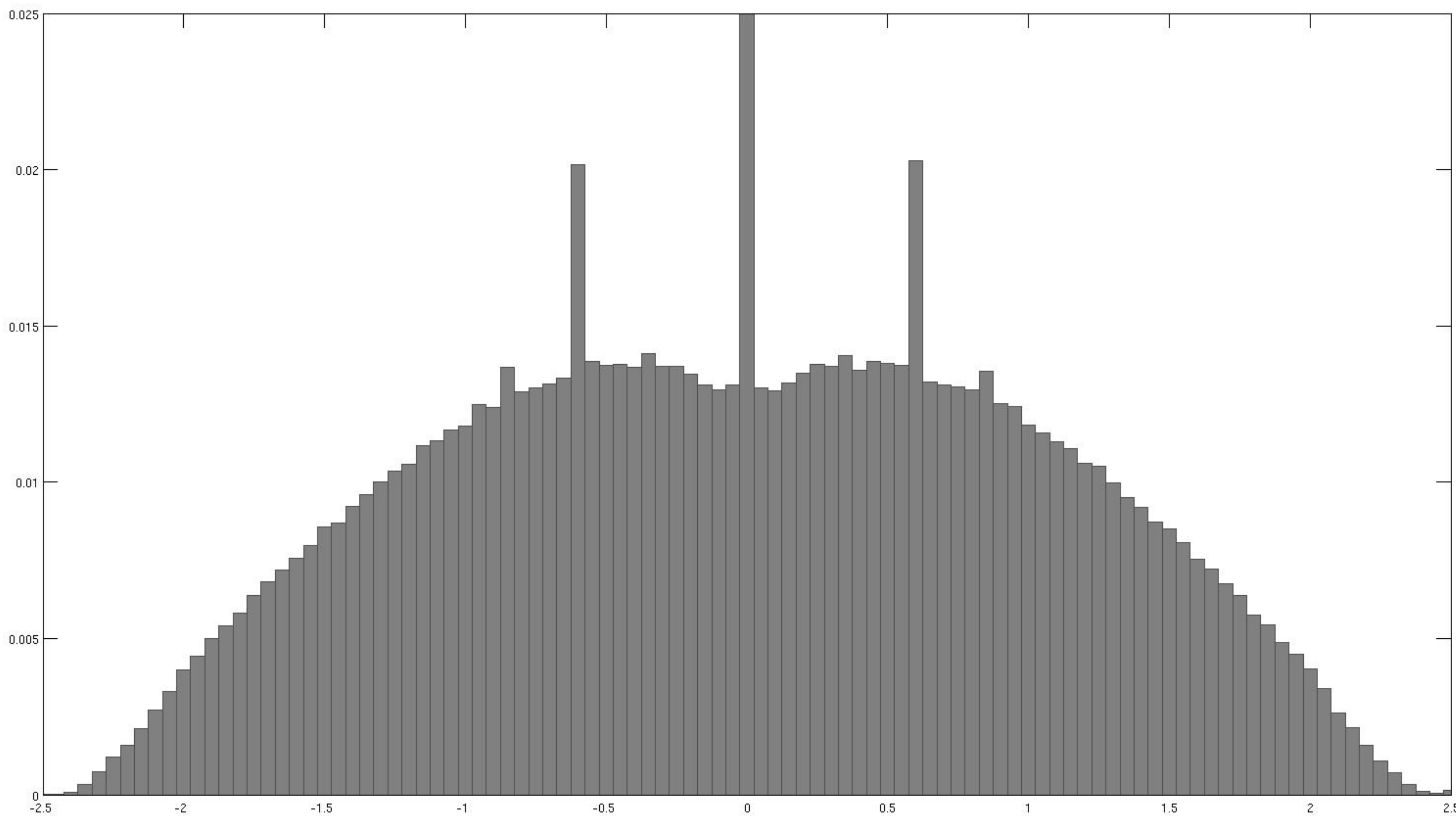
Numerical simulations on diluted graphs with 5000 vertices

$$c = 2,8$$



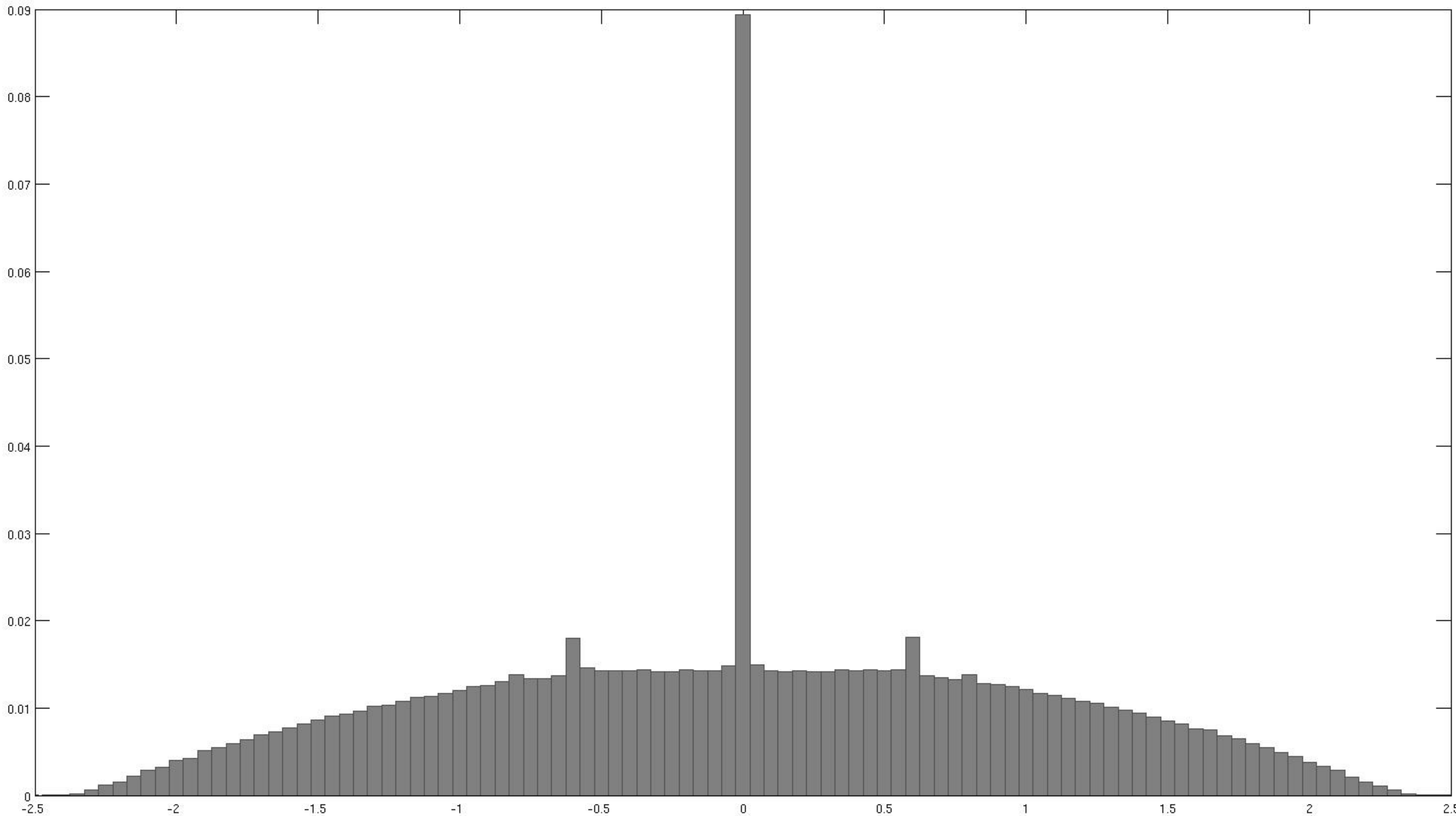
Numerical simulations on diluted graphs with 5000 vertices

$c = 2,8$ (zoomed in)



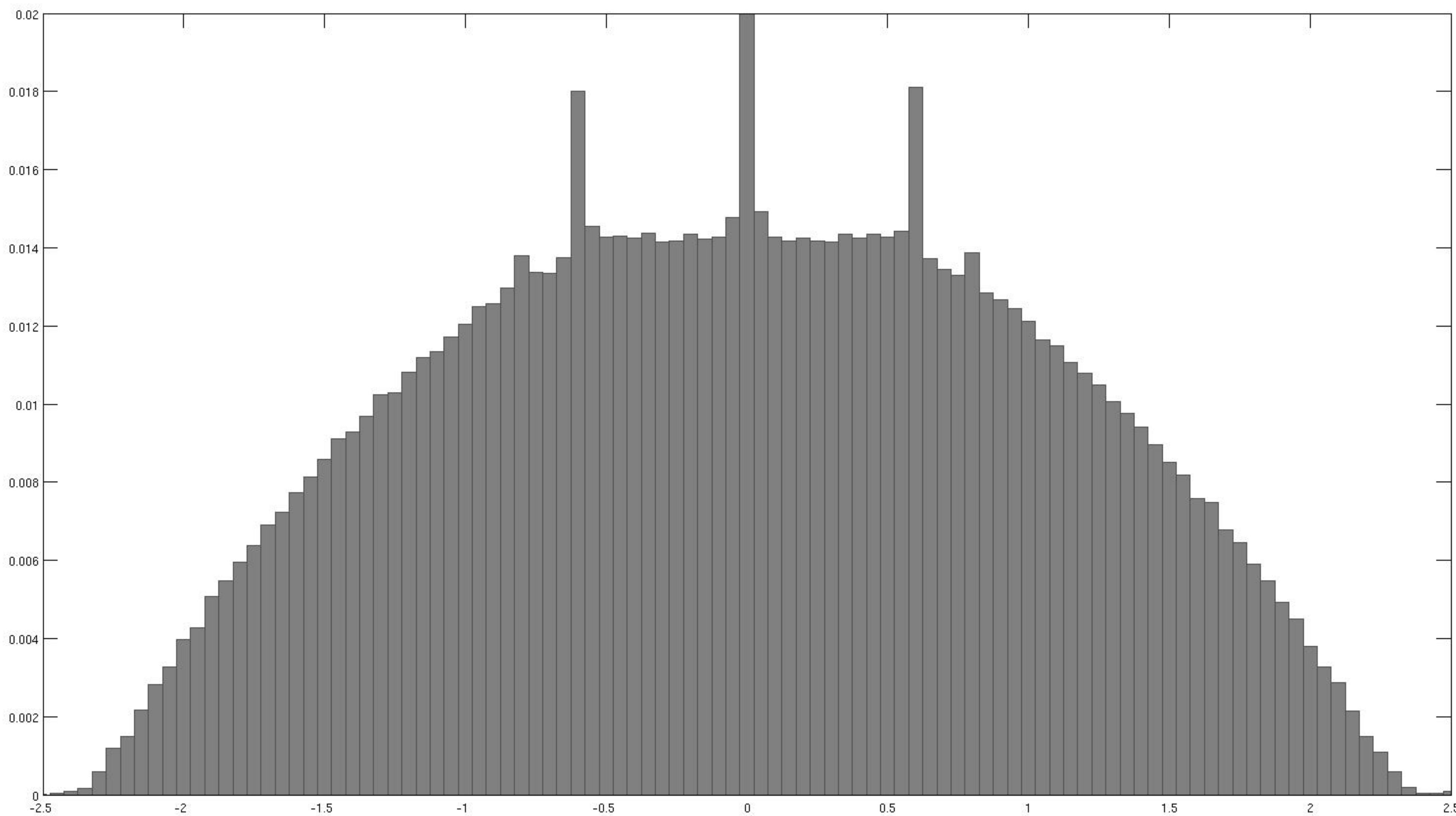
Numerical simulations on diluted graphs with 5000 vertices

$$c = 3$$



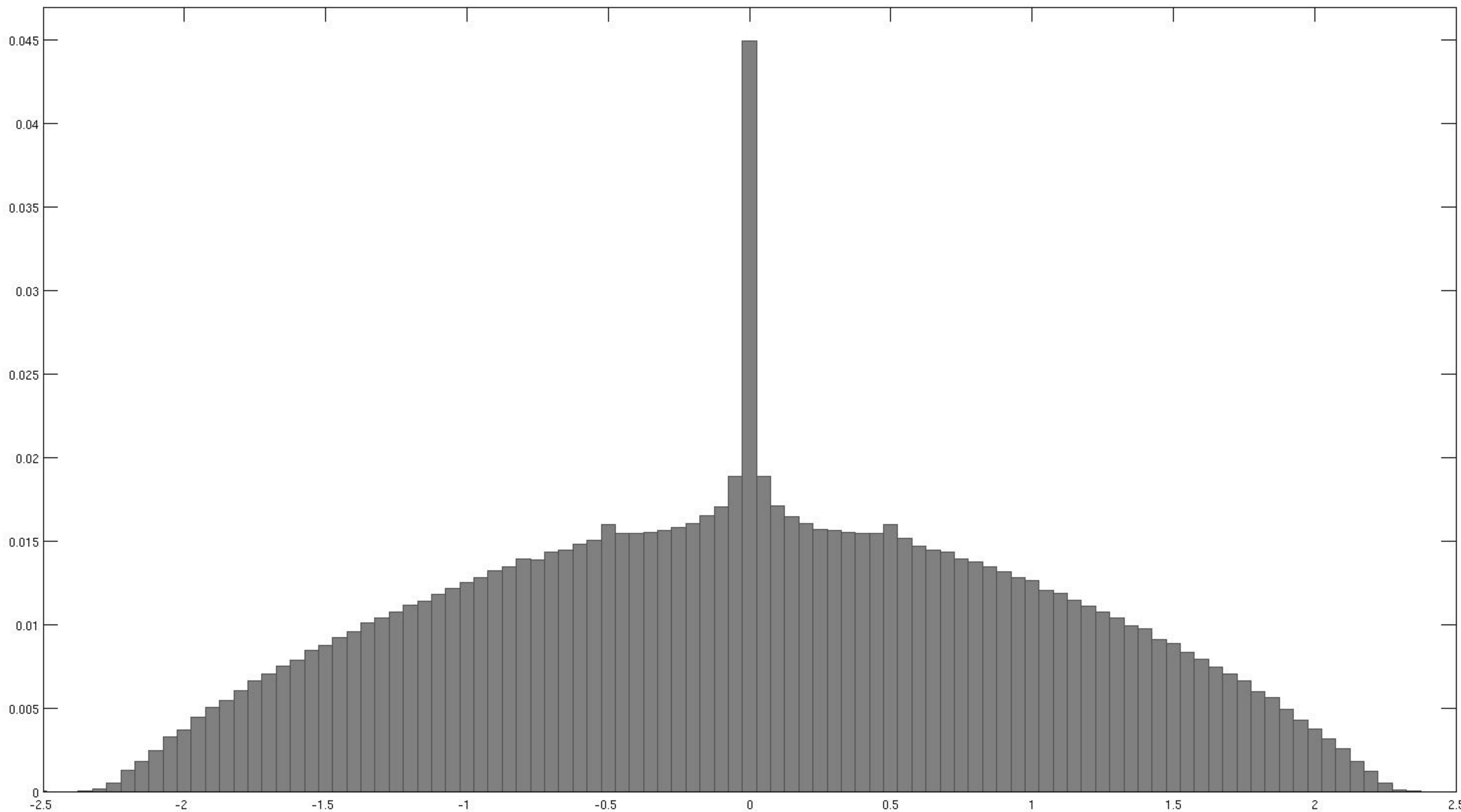
Numerical simulations on diluted graphs with 5000 vertices

$c = 3$ (zoomed in)



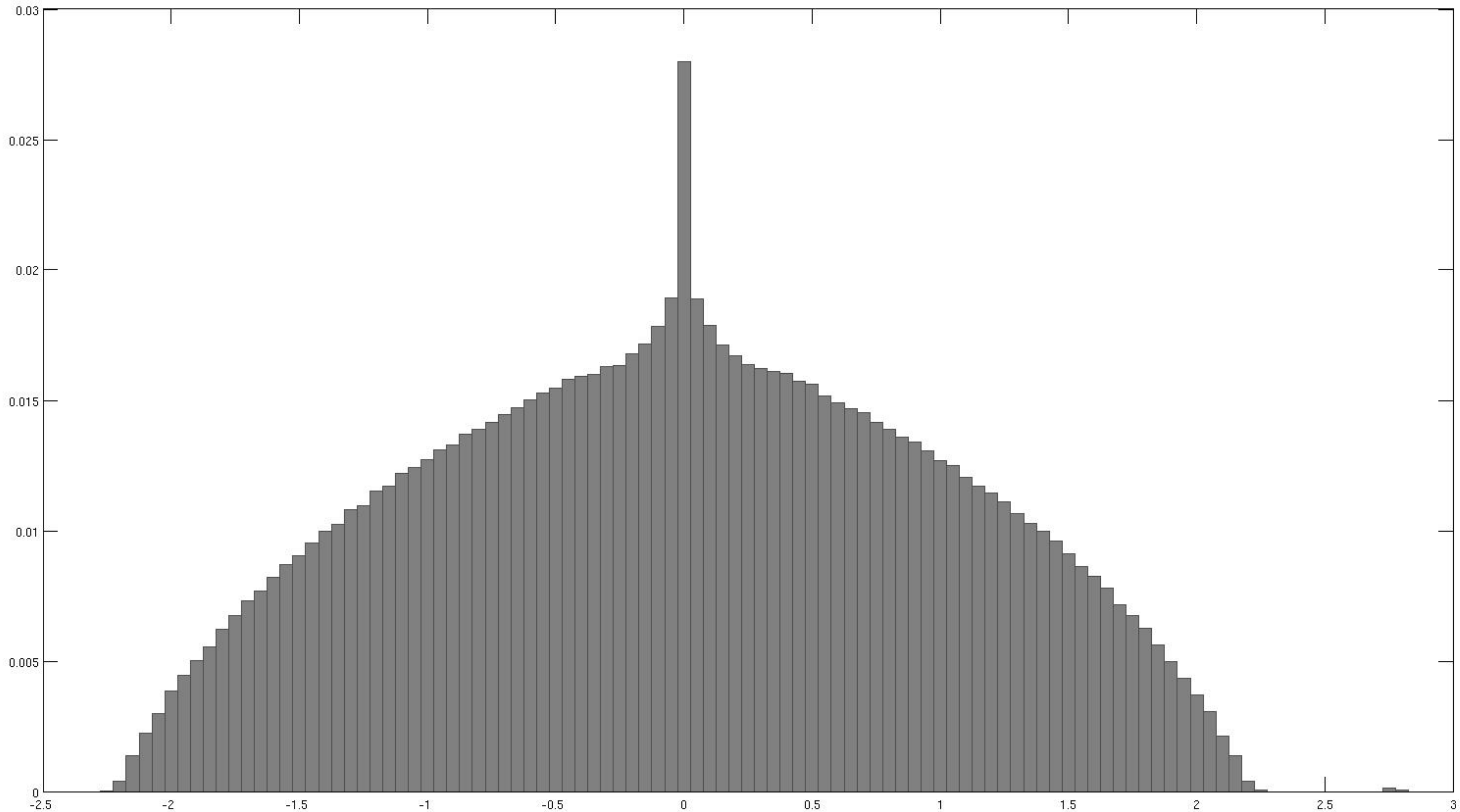
Numerical simulations on diluted graphs with 5000 vertices

$$c = 4$$



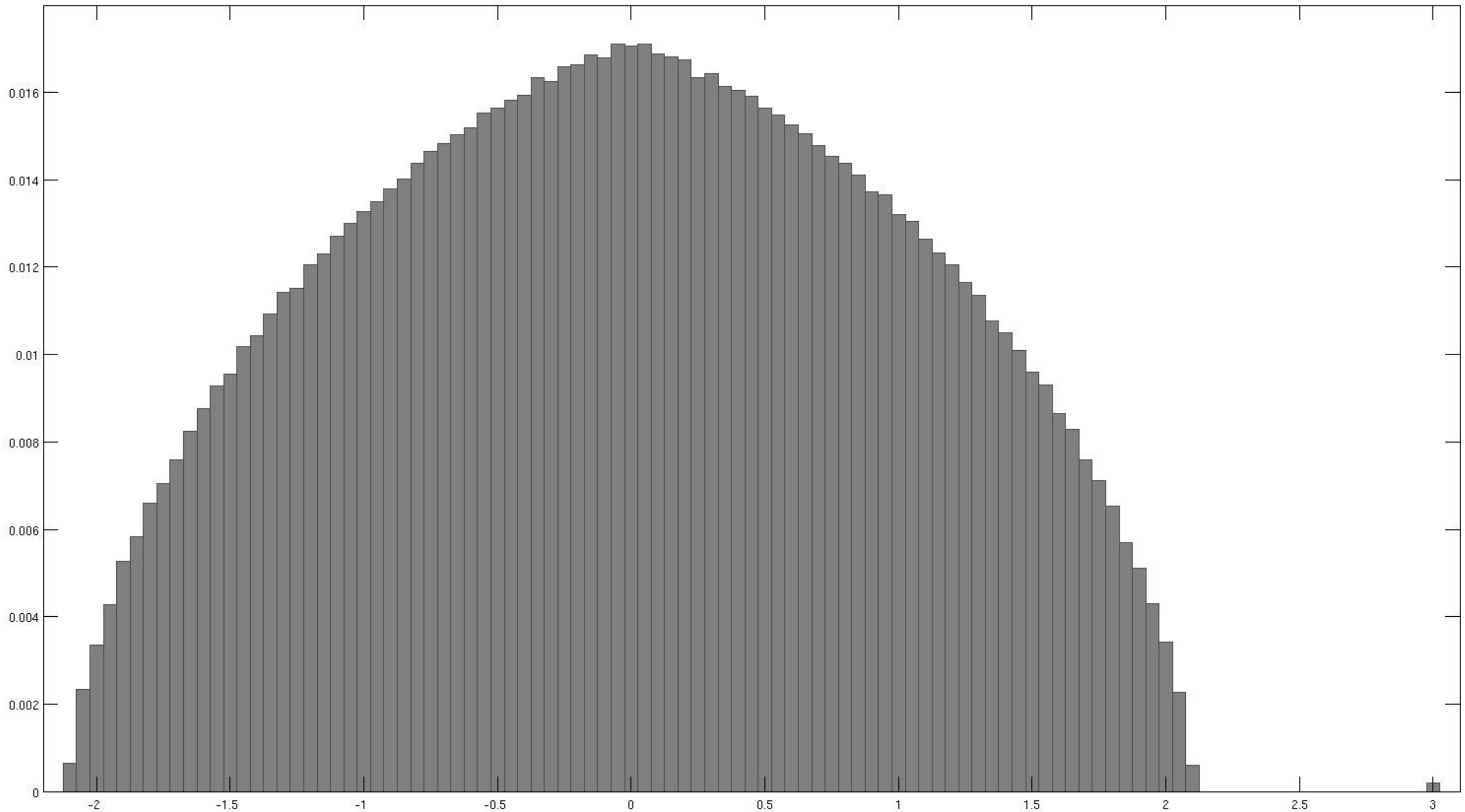
Numerical simulations on diluted graphs with 5000 vertices

$$c = 5$$



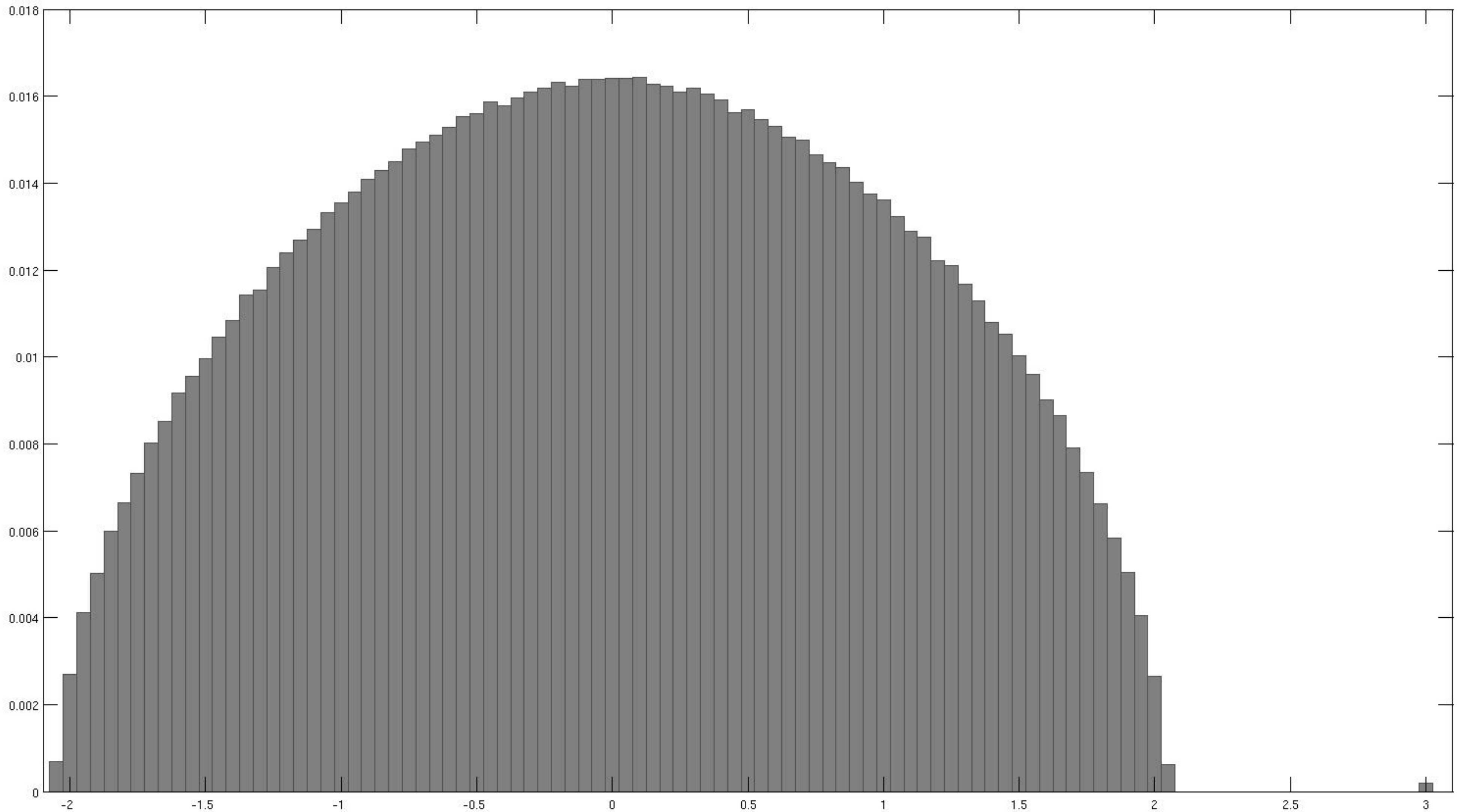
Numerical simulations on diluted graphs with 5000 vertices

$$c = 10$$



Numerical simulations on diluted graphs with 5000 vertices

$c = 20$



État de l'art

$$\mu_n^c = \frac{1}{n} \sum_{\lambda \in \mathrm{Sp}(c^{-1/2}A)} \delta_\lambda$$: empirical spectral distribution of $G(n, c/n)$

État de l'art

$$\mu_n^c = \frac{1}{n} \sum_{\lambda \in \mathrm{Sp}(c^{-1/2}A)} \delta_\lambda$$: empirical spectral distribution of $G(n, c/n)$

Fact : as $n \to \infty$, μ_n^c converges weakly to a probability measure μ^c

État de l'art

$$\mu_n^c = \frac{1}{n} \sum_{\lambda \in \mathrm{Sp}(c^{-1/2}A)} \delta_\lambda$$: empirical spectral distribution of $G(n, c/n)$

Fact : as $n \to \infty$, μ_n^c converges weakly to a probability measure μ^c

Known properties of μ^c:

- if $c \to \infty$, μ^c converges weakly to Wigner semi-circle law
- unbounded support
- dense set of atoms

État de l'art

$$\mu_n^c = \frac{1}{n} \sum_{\lambda \in \mathrm{Sp}(c^{-1/2}A)} \delta_\lambda$$: empirical spectral distribution of $G(n, c/n)$

Fact : as $n \to \infty$, μ_n^c converges weakly to a probability measure μ^c

Known properties of μ^c:

- if $c \to \infty$, μ^c converges weakly to Wigner semi-circle law
- unbounded support
- dense set of atoms
- $\mu^c(\{0\})$ known explicitly [Bordenave, Lelarge, Salez 2011]

État de l'art

$$\mu_n^c = \frac{1}{n} \sum_{\lambda \in \mathrm{Sp}(c^{-1/2}A)} \delta_\lambda$$: empirical spectral distribution of $G(n, c/n)$

Fact : as $n \to \infty$, μ_n^c converges weakly to a probability measure μ^c

Known properties of μ^c:

- if $c \to \infty$, μ^c converges weakly to Wigner semi-circle law
- unbounded support
- dense set of atoms
- $\mu^c(\{0\})$ known explicitly [Bordenave, Lelarge, Salez 2011]
- μ^c is not purely atomic *iif* $c > 1$ [Bordenave, Sen, Virág 2013]

Asymptotic expansion of the spectrum

Asymptotic expansion of the spectrum

If μ is a (signed) mesure and $\int |x|^k |d\mu(x)| < \infty$, denote $m_k(\mu) = \int x^k d\mu(x)$

Asymptotic expansion of the spectrum

If μ is a (signed) mesure and $\int |x|^k |d\mu(x)| < \infty$, denote $m_k(\mu) = \int x^k d\mu(x)$

Theorem: For every $k \geq 0$ and as $c \to \infty$

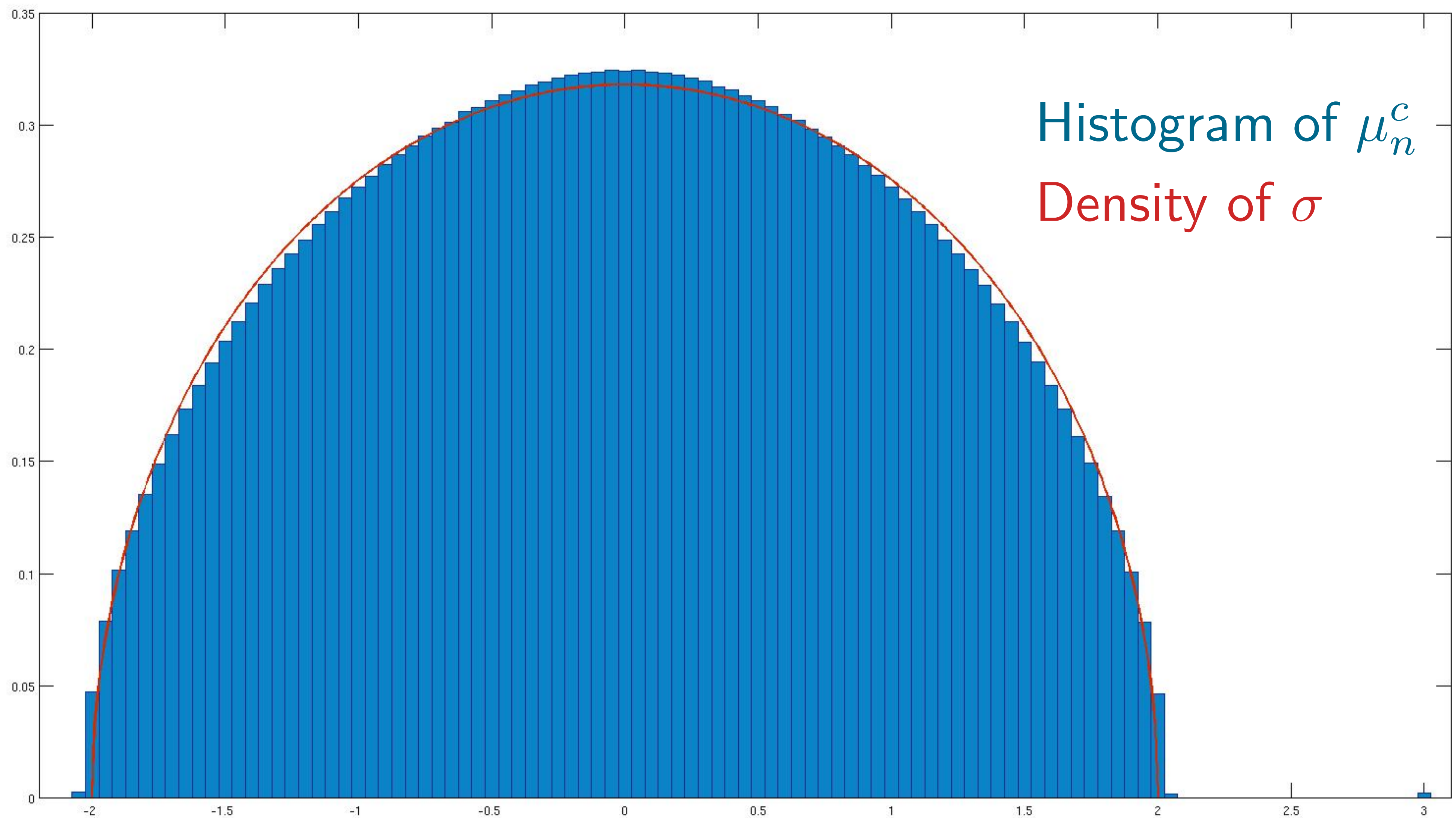
$$m_k(\mu^c) = m_k(\sigma) + \frac{1}{c} m_k(\sigma^{\{1\}}) + o\left(\frac{1}{c}\right)$$

where σ is the semi-circle law having density $\frac{1}{2\pi}\sqrt{4 - x^2}\mathbf{1}_{|x|<2}$
and $\sigma^{\{1\}}$ is a measure with total mass 0 and density

$$\frac{1}{2\pi} \frac{x^4 - 4x^2 + 2}{\sqrt{4 - x^2}} \mathbf{1}_{|x|<2}.$$

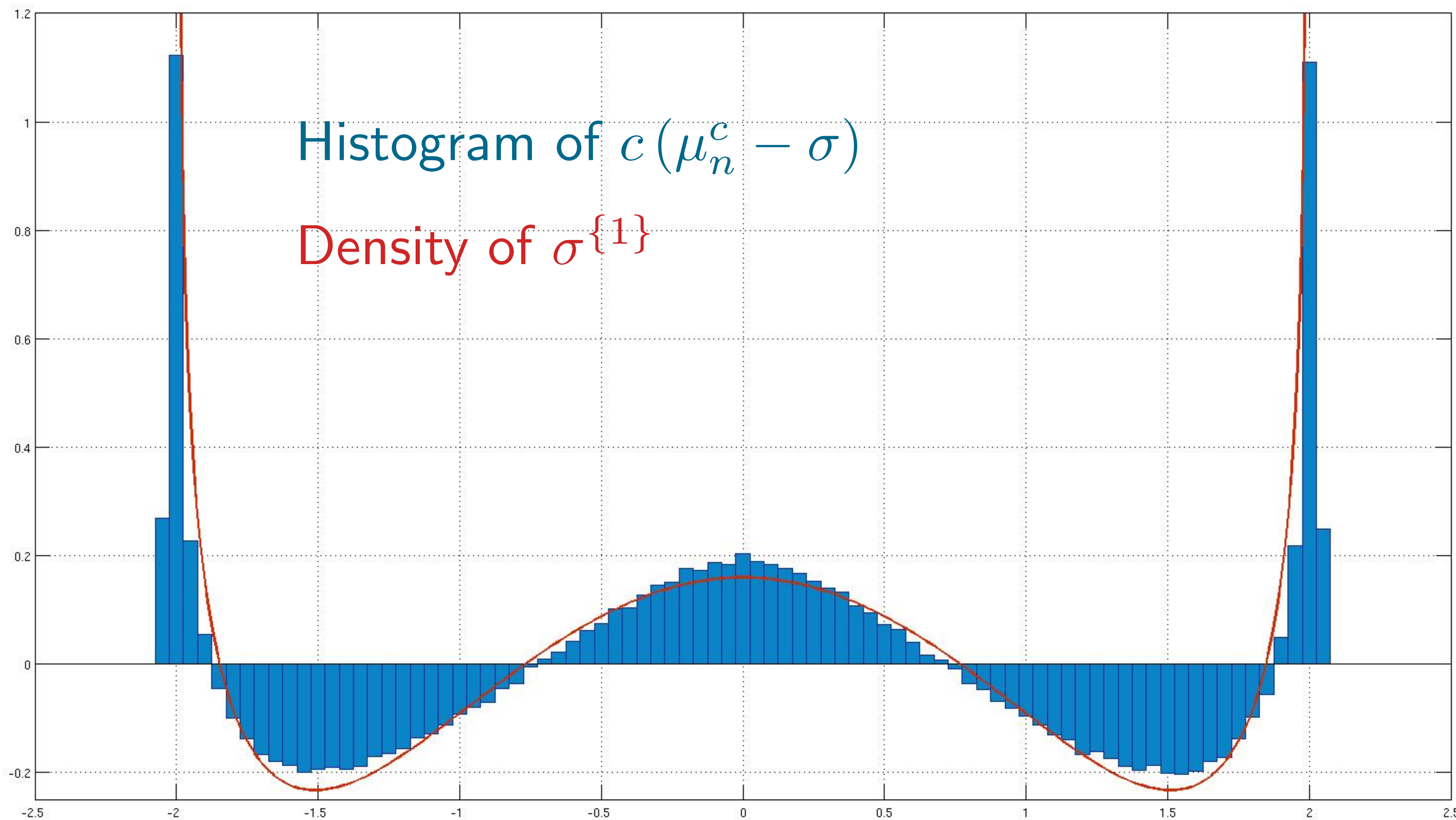
Asymptotic expansion of the spectrum – numerical simulations

100 matrices of size 10000 with $c = 20$



Asymptotic expansion of the spectrum – numerical simulations

100 matrices of size 10000 with $c = 20$



Asymptotic expansion of the spectrum: second order (I)

Proposition: For every $k \geqslant 0$ we have the following asymtotic expansion in c:

$$m_k(\mu^c) = m_k(\sigma) + \frac{1}{c} m_k(\sigma^{\{1\}}) + \frac{1}{c^2} d_k + o\left(\frac{1}{c^2}\right)$$

where the numbers d_k are **NOT** the moments of a measure!

Asymptotic expansion of the spectrum: second order (I)

Proposition: For every $k \geqslant 0$ we have the following asymtotic expansion in c:

$$m_k(\mu^c) = m_k(\sigma) + \frac{1}{c} m_k(\sigma^{\{1\}}) + \frac{1}{c^2} d_k + o\left(\frac{1}{c^2}\right)$$

where the numbers d_k are **NOT** the moments of a measure!

→ *The asymptotic expansion must take into account the fact that*

$$\mu^c\left(\mathbb{R} \setminus [-2;2]\right) = \mathcal{O}\left(\frac{1}{c^2}\right).$$

Asymptotic expansion of the spectrum: second order (I)

Proposition: For every $k \geqslant 0$ we have the following asymtotic expansion in c:

$$m_k(\mu^c) = m_k(\sigma) + \frac{1}{c} m_k(\sigma^{\{1\}}) + \frac{1}{c^2} d_k + o\left(\frac{1}{c^2}\right)$$

where the numbers d_k are **NOT** the moments of a measure!

→ *The asymptotic expansion must take into account the fact that*

$$\mu^c\left(\mathbb{R} \setminus [-2;2]\right) = \mathcal{O}\left(\frac{1}{c^2}\right).$$

Dilation operator Λ_α for measures defined by $\Lambda_\alpha(\mu)(A) = \mu\left(A/\alpha\right)$ for a measure μ and a Borel set A.

Asymptotic expansion of the spectrum: second order (I)

Proposition: For every $k \geqslant 0$ we have the following asymtotic expansion in c:

$$m_k(\mu^c) = m_k(\sigma) + \frac{1}{c} m_k(\sigma^{\{1\}}) + \frac{1}{c^2} d_k + o\left(\frac{1}{c^2}\right)$$

where the numbers d_k are **NOT** the moments of a measure!

→ *The asymptotic expansion must take into account the fact that*

$$\mu^c\left(\mathbb{R} \setminus [-2;2]\right) = \mathcal{O}\left(\frac{1}{c^2}\right).$$

Dilation operator Λ_α for measures defined by $\Lambda_\alpha(\mu)(A) = \mu\left(A/\alpha\right)$ for a measure μ and a Borel set A.

For example, $\Lambda_\alpha(\sigma)$ is supported on $[-2\alpha; 2\alpha]$.

Asymptotic expansion of the spectrum: second order (II)

Theorem: For every $k \geq 0$ and as $c \to \infty$

$$m_k(\mu^c) = m_k\left(\Lambda_{1+\frac{1}{2c}}\left(\sigma + \frac{1}{c}\hat{\sigma}^{\{1\}} + \frac{1}{c^2}\hat{\sigma}^{\{2\}}\right)\right) + o\left(\frac{1}{c^2}\right)$$

where $\hat{\sigma}^{\{1\}}$ is a measure with null total mass and density

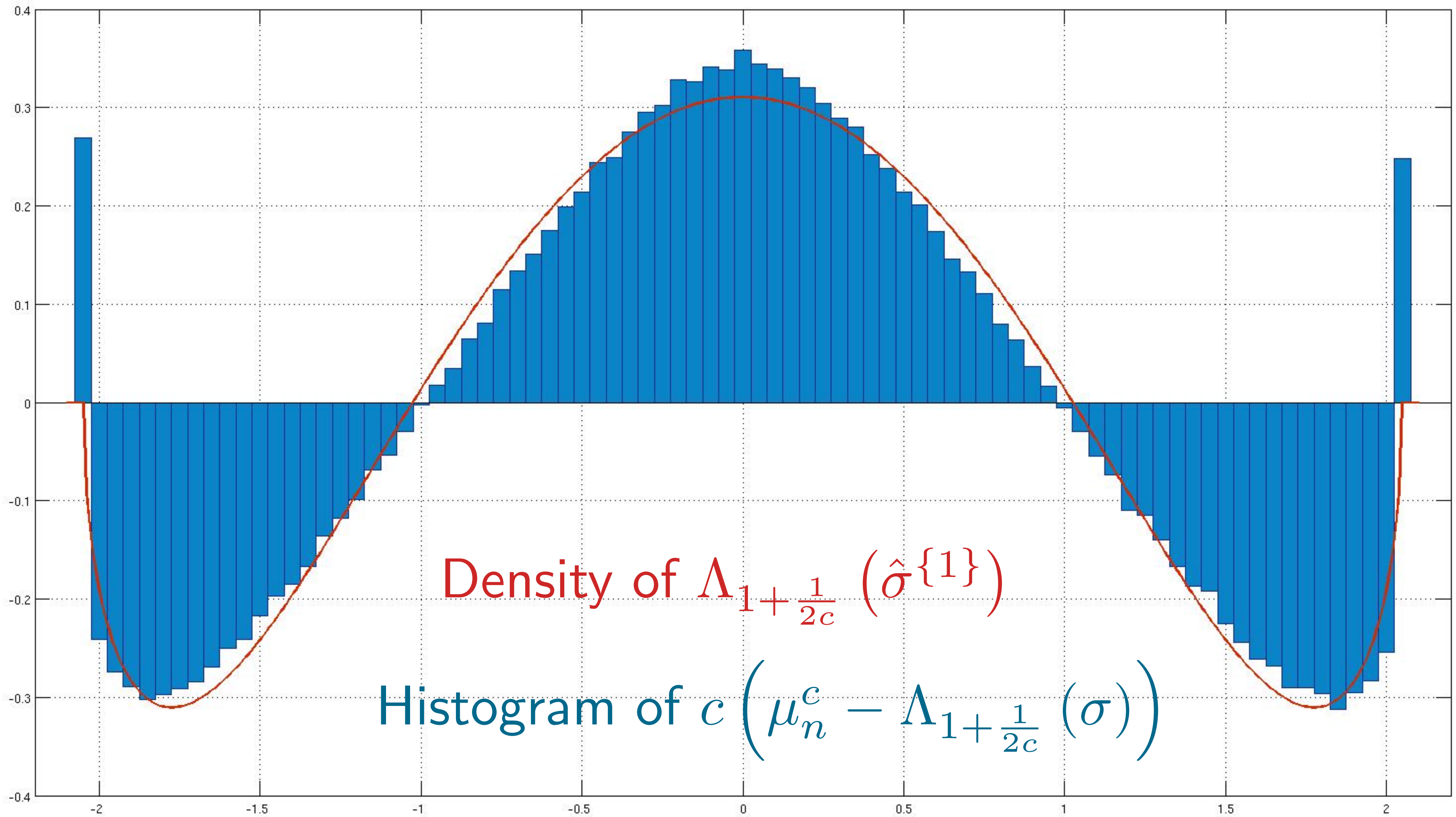
$$-\frac{x^4 - 5x^2 + 4}{2\pi\sqrt{4 - x^2}}\mathbf{1}_{|x|<2}$$

and where $\hat{\sigma}^{\{2\}}$ is a measure with null total mass and density

$$-\frac{2x^8 - 17x^6 + 46x^4 - \frac{325}{8}x^2 + \frac{21}{4}}{\pi\sqrt{4 - x^2}}\mathbf{1}_{|x|<2}.$$

Second order – numerical simulations

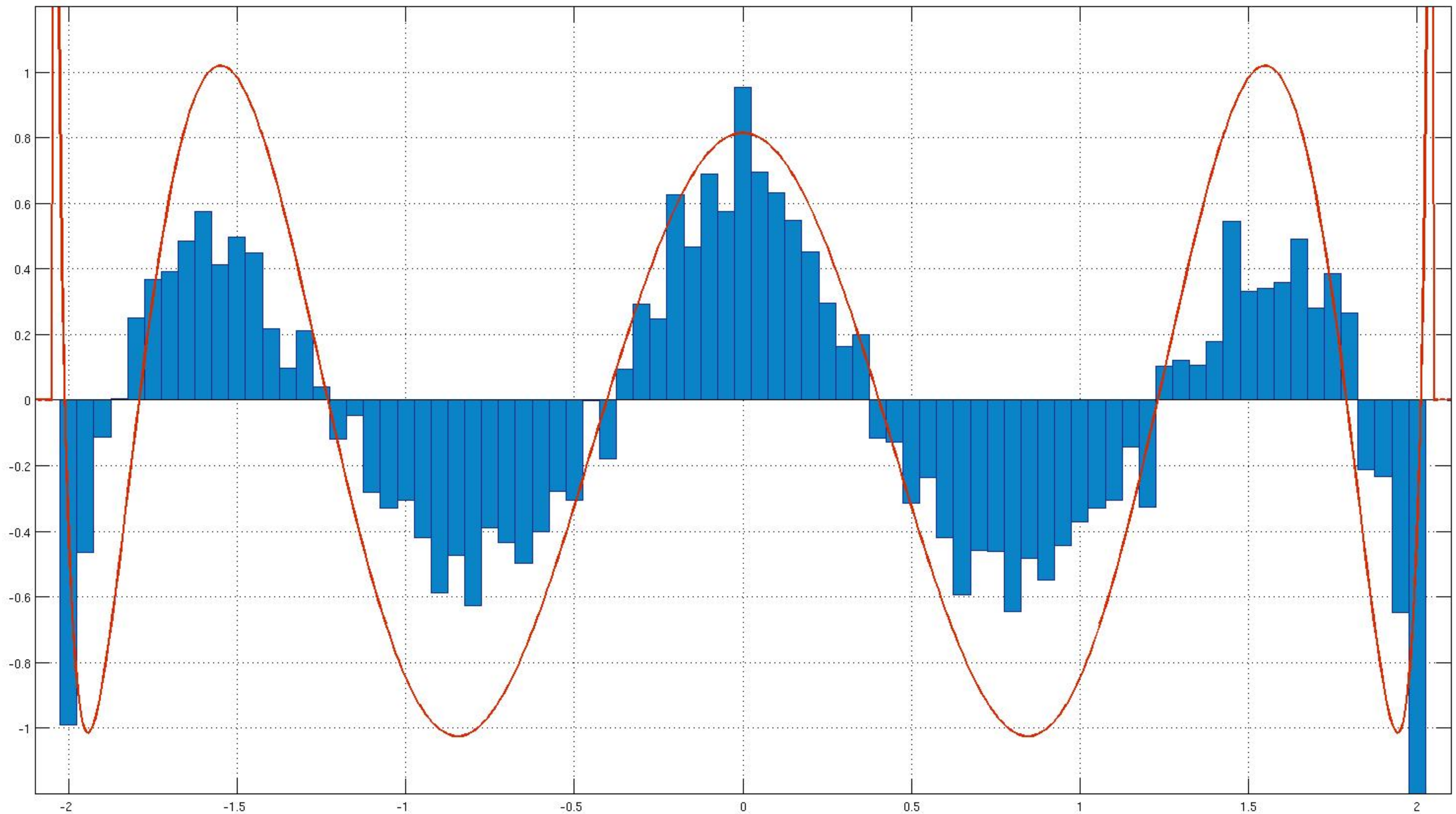
100 matrices of size 10000 with $c = 20$



Second order – numerical simulations

100 matrices of size 10000 with $c = 20$

Histogram of $c^2 \left(\mu_n^c - \Lambda_{1+\frac{1}{2c}} \left(\sigma + \frac{1}{c} \hat{\sigma}^{\{1\}} \right) \right)$ Density of $\Lambda_{1+\frac{1}{2c}} \left(\hat{\sigma}^{\{2\}} \right)$



Edge of the Spectrum

$$m_k(\mu^c) = m_k\left(\Lambda_{1+\frac{1}{2c}}\left(\sigma + \frac{1}{c}\hat{\sigma}^{\{1\}} + \frac{1}{c^2}\hat{\sigma}^{\{2\}}\right)\right) + o\left(\frac{1}{c^2}\right)$$

The measure on the right hand side is supported on $[-2-1/c; 2+1/c]$.

Edge of the Spectrum

$$m_k(\mu^c) = m_k\left(\Lambda_{1+\frac{1}{2c}}\left(\sigma + \frac{1}{c}\hat{\sigma}^{\{1\}} + \frac{1}{c^2}\hat{\sigma}^{\{2\}}\right)\right) + o\left(\frac{1}{c^2}\right)$$

The measure on the right hand side is supported on $[-2-1/c; 2+1/c]$.

→ This suggests that for $\varepsilon > 0$, as $c \to \infty$,

$$\mu^c\left(\left]-\infty; -2-\frac{1+\varepsilon}{c}\right] \cup \left[2+\frac{1+\varepsilon}{c}; +\infty\right[\right) = o\left(\frac{1}{c^2}\right).$$

Edge of the Spectrum

$$m_k(\mu^c) = m_k\left(\Lambda_{1+\frac{1}{2c}}\left(\sigma + \frac{1}{c}\hat{\sigma}^{\{1\}} + \frac{1}{c^2}\hat{\sigma}^{\{2\}}\right)\right) + o\left(\frac{1}{c^2}\right)$$

The measure on the right hand side is supported on $[-2 - 1/c; 2 + 1/c]$.

→ This suggests that for $\varepsilon > 0$, as $c \to \infty$,

$$\mu^c\left(\left]-\infty; -2 - \frac{1+\varepsilon}{c}\right] \cup \left[2 + \frac{1+\varepsilon}{c}; +\infty\right[\right) = o\left(\frac{1}{c^2}\right).$$

Thank you!