Free Probability and Random Graphs

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8 janvier 2016

Monday in Mira's talk we met the following situation

$$H_N = X_N + A_N, \quad \text{where}$$

- X_N GOE matrix,
- A_N Hermitian matrix independent of X_N.

Conclusion : The limiting empirical spectral distribution (e.s.d.) ν of H_N is the "free convolution" of the semicircular law σ and of the limiting e.s.d. μ of A_N.

$$\text{Operation} : (\sigma, \mu) \mapsto \nu = \sigma \boxplus \mu$$

The notion below this convergence is the *"asymptotic free independence"* of X_N and A_N. It gives an algebraic rule that characterizes the limiting e.s.d. of any Hermitian matrix $H_N = P(X_N, A_N)$.

Purpose of the talk : for adjacency matrices of graphs, this phenomenon is

- not always true for large sparse graphs (order one degree),
- and still true for certain large dense graphs (degree $\to \infty$).

Erdös-Rényi (E.R.) and Uniform Regular Graphs (U.R.G.) will be investigated and it is shown that they have different behavior in the sparse case. But we do not restrict to these ensembles.

Our analysis is based on

- an extension of algebraic framework of Free Probability [Voiculescu '86], in particular *New notions of Convergence and Asymptotic Independence* for large matrices [M.11+],
- a comparison of this new setting with Weak Local Convergence [Benjamini-Schramm'01] and inspiration from the Convergence of Graphons [Lovász-Szegedy'06], contained in [M.11+] and [M. Péché 14] respectivelyy.

Presentation :

1. General notions of free probability
2. New notion of convergence and of asymptotic independence
3. Sparse case
4. Dense case

Non Commutative Probability for Random Matrices

The Non Commutative Distribution of a family $\mathbf{A}_N = (A_j)_{j\in J}$ of random matrices is the map

$$\Phi_{\mathbf{A}_N} : P \mapsto \mathbb{E}\big[\tfrac{1}{N}\mathrm{Tr} P(\mathbf{A}_N)\big]$$

defined for non commutative polynomials P in the A_j and A_j^*, $j \in J$.

The Convergence in N. C. Distribution of $\mathbf{A}_N$ is the convergence of $\Phi_{\mathbf{A}_N}(P)$ for each P.

Let $H_N = P(\mathbf{A}_N)$ be Hermitian (P is fixed) and denote $\mathcal{L}_{H_N} = \mathbb{E}\big[\frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}\big]$ its mean eigenvalues distribution. Then for any polynomial Q

$$\mathcal{L}_{H_N}(Q) = \mathbb{E}\Big[\tfrac{1}{N}\mathrm{Tr} Q(H_N)\Big] = \mathbb{E}\tfrac{1}{N}\mathrm{Tr} Q\big(P(\mathbf{A}_N)\big) = \Phi_{\mathbf{A}_N}\big(Q(P)\big)$$

So point wise convergence of $\Phi_{\mathbf{A}_N}$ implies convergence in moments of H_N. It also implies the convergence in moments of the mean e.s.d. of every block matrices whose blocks are polynomials in $\mathbf{A}_N$.

Definition

The families of matrices $\mathbf{A}_N^{(1)}, \dots, \mathbf{A}_N^{(L)}$ *are Asymptotically freely independent iff*

1. *each family* $\mathbf{A}_N^{(\ell)}$ *converges in N.C. Distribution,*
2. $\forall K \geq 1$*,* $\forall P_1, \dots, P_K$ *non commutative polynomials in the matrices and their adjoint*

$$\Phi\Big[P_1(\mathbf{A}_N^{(i_1)}) \dots P_K(\mathbf{A}_N^{(i_K)})\ \Big] \underset{N\to\infty}{\longrightarrow} 0$$

as soon as $i_1 \neq i_2 \neq \dots \neq i_K$ *and* $\Phi\big[P_k(\mathbf{A}_N^{(i_k)})\ \big] \underset{N\to\infty}{\longrightarrow} 0$ *for each* $k = 1, \dots, K$*.*

Knowing the limiting N.C. distributions of each $\mathbf{A}_N^{(\ell)}$, the above rule characterizes the limiting N.C. distribution of $\mathbf{A}_N^{(1)}, \dots, \mathbf{A}_N^{(L)}$.

Theorem (Voiculescu (91), Collins and Śniady (04))

$\mathbf{A}_N^{(1)}, \dots, \mathbf{A}_N^{(L)}$ *independent families of random matrices such that*

1. *each family (except possibly one) is unitarily invariant,*
2. *each family converges in N.C. Distribution,*
3. *+ technical condition, say* $\mathbf{A}_N^{(\ell)} = U\tilde{\mathbf{A}}_N^{(\ell)}U^*$ *with* $\tilde{\mathbf{A}}_N^{(\ell)}$ *deterministic.*

Then $\mathbf{A}_N^{(1)}, \dots, \mathbf{A}_N^{(L)}$ *are asymptotically freely independent.*

The conclusion remains valid if one family consists in independent Wigner matrices and independent matrices according to the Haar measure on the unitary group.

Asymptotic free independence is also true for other models, e.g. for a family independent uniform random permutation matrices $\mathbf{V}_N = (V_N^{(1)}, \dots, V_N^{(L)})$ [Nica'93]. So adjacency matrices $A_N^{(1)}, \dots, A_N^{(L)}$ of independent $2d_k$-regular graphs (sums of uniform permutations matrices as long as their transpose) are asymptotically freely independent.

Remark that a family of independent unitary matrices distributed according to the Haar measure $\mathbf{U}_N$ and is a family of independent uniform permutation matrices $\mathbf{V}_N$ have the same limiting N.C. distribution. Hence Hermitian matrices of the form $H_N = P(\mathbf{U}_N)$ and $\tilde{H}_N = P(\mathbf{V}_N)$ have the same limiting e.s.d.

Theorem

1. *[Pimsner-Shlyakhtenko'96, Guionnet-Shlyakhtenko'09] The support of the limiting e.s.d. of $H_N = P(\mathbf{U}_N)$ is connected. So the same hold for $\tilde{H}_N = P(\mathbf{V}_N)$.*
2. *[M. Collins'14] For N large enough no eigenvalue of $H_N = P(\mathbf{U}_N)$ lies outside a small neighborhood of the limiting support, in particular*

$$\| U_N^{(1)} + U_N^{(1)*} + \cdots + U_N^{(L)} + U_N^{(L)*} \| \underset{N\to\infty}{\longrightarrow} 2\sqrt{L-1}, \text{ a.s.}$$

The asymptotic free independence of adjacency matrices of large regular graphs with large degree was not considered before in the theory. It is also true but we shall need the tools introduced in the next section.

For the adjacency matrix $A(d_N)$ of an E.R. graphs with parameter $\frac{d_N}{N}$, we denote its standard version $M(d_N) = \frac{A(d_N) - \frac{d_N}{N} J_N}{\sqrt{d_N(1-\frac{d_N}{N})}}$

Proposition (Ryan'98)

Let $\mathbf{M}_N = \big(M(d_N^{(1)}), \dots , M(d_N^{(L)}) \big)$ be independent standardized E.R. matrices.

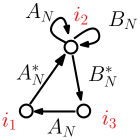
1. *If $d_N^{(\ell)} \underset{N\to\infty}{\longrightarrow} \infty$ for each ℓ, then the matrices of $\mathbf{M}_N$ are asymptotically freely independent.*
2. *If $d_N^{(\ell)} \underset{N\to\infty}{\longrightarrow} d^{(\ell)} > 0$ for each ℓ, this is not true.*

Let $\mathbf{Y}_N$ be a family of deterministic matrices, converging in N.C. Distr.
For $d_N \underset{N\to\infty}{\longrightarrow} \infty$, it is also true that M_N and $\mathbf{Y}_N$ are asymptotically free.
For $d_N \underset{N\to\infty}{\longrightarrow} d > 0$, the possible N.C. Distr. of M_N and $\mathbf{Y}_N$ depend on much more than the limiting N.C. Dist. of $\mathbf{Y}_N$.

Generalization of N.C. Distributions and new concept of asymptotic independence

The Distribution of Traffics of a family $\mathbf{A}_N = (A_j)_{j\in J}$ of random matrices is the map $\tau_{\mathbf{A}_N} : T \mapsto \mathbb{E}\big[\frac{1}{N}\mathrm{Tr}\, T(\mathbf{A}_N)\big]$ defined for any finite connected graph T whose edges are labeled by the A_j and A_j^*, $j \in J$ as follow : denoting $T = (V, E, \gamma, \varepsilon)$ where $\gamma : E \to J$ and $\varepsilon : E \to \{1, *\}$,

$$\mathrm{Tr}\, T(\mathbf{A}_N) = \sum_{\phi : V \to \{1,\dots,N\}} \prod_{e=(v,w)\in E} A_{\gamma(e)}^{\varepsilon(e)}\big(\phi(v), \phi(w)\big)$$



Ex : $\mathbb{E}\frac{1}{N}\sum_{i_1,i_2,i_3} A_N^*(i_1,i_2)A_N(i_2,i_2)B_N(i_2,i_2)B_N^*(i_2,i_3)A_N(i_3,i_1)$

The Convergence in distribution of traffics of $\mathbf{A}_N$ is the convergence of $\tau_{\mathbf{A}_N}(T)$ for each T. If T consists in a simple oriented cycle, then $\tau_{\mathbf{A}_N}(T) = \mathbb{E}[\frac{1}{N}\mathrm{Tr} P(\mathbf{A}_N)] = \Phi_{\mathbf{A}_N}(P)$ where P is the monomials consisting in the product of the matrices along the cycle.

Theorem (M. 11+)

$\mathbf{A}_N^{(1)}, \dots, \mathbf{A}_N^{(L)}$ *independent families of random matrices such that*

1. *each family (except possibly one) is permutation invariant,*
2. *each family converges in distribution of traffics,*
3. *+ technical condition, say the factorization property : for any ℓ and any $T_1, \dots, T_p$*

$$\lim_{N\to\infty} \mathbb{E}\big[\prod_{i=1}^p \frac{1}{N}\mathrm{Tr}\, T_i(\mathbf{A}_N^{(\ell)})\big] = \prod_{i=1}^p \lim_{N\to\infty} \mathbb{E}\big[\frac{1}{N}\mathrm{Tr}\, T_i(\mathbf{A}_N^{(\ell)})\big].$$

Then $(\mathbf{A}_N^{(1)}, \dots, \mathbf{A}_N^{(L)})$ converges in distribution of traffics (and so in N.C. distribution). The families are said to be asymptotically traffic independent. An explicit formula exists such that, knowing the limiting distribution of traffics of each $\mathbf{A}_N^{(\ell)}$, it characterizes the limiting distribution of $\mathbf{A}_N^{(1)}, \dots, \mathbf{A}_N^{(L)}$.

For each finite connected graph labeled by variables and their adjoint $T = (V, E, \gamma, \varepsilon)$ we introduce the injective trace

$$\mathrm{Tr}^0\big[T(\mathbf{A}_N)\big] = \sum_{\substack{\phi: V \to [N] \\ \text{injective}}} \prod_{e=(v,w)\in E} A_{\gamma(e)}^{\varepsilon(e)}\big(\phi(v), \phi(w)\big).$$

For a partition π of the vertex set V, denote by T^π the quotient graph. Then for each $T = (V, E, \gamma, \varepsilon)$,

$$\mathrm{Tr}\big[T(\mathbf{A}_N)\big] = \sum_{\pi \in \mathcal{P}(V)} \mathrm{Tr}^0\big[T^\pi(\mathbf{A}_N)\big],$$

and one can also express Tr^0 in terms of Tr.

Moreover, if $\mathbf{A}_N$ is permutation invariant, then $\delta_N^0[T(\mathbf{A}_N)] := \mathbb{E}\Big[\prod_{e=(v,w)\in E} A_{\gamma(e)}^{\varepsilon(e)}\big(\phi(v), \phi(w)\big)\Big]$ does not depend on ϕ, so that

$$\mathbb{E}\frac{1}{N}\mathrm{Tr}^0\big[T(\mathbf{A}_N)\big] = N^{V-1}\big(1 + o(1)\big)\delta_N^0[T(\mathbf{A}_N)]$$

.

Proposition (M'11+,Cesbron-Dahlqvist-M.'16)

Let $A_N^{(1)}, \dots, A_N^{(L)}$ be asymptotically traffic independent. Then they are asymptotically freely independent if and only if, with $\Phi := \lim_{N\to\infty} \mathbb{E}\big[\frac{1}{N}\mathrm{Tr}\,\cdot\,\big]$, for each ℓ and any polynomials P_1, P_2

$$\Phi\big[\, P_1(A_N^{(\ell)}) \circ P_2(A_N^{(\ell)})\big] - \Phi\big[P_1(A_N^{(\ell)})\big] \times \Phi\big[P_2(A_N^{(\ell)})\big] = 0.$$

Example : for diagonal matrices and adjacency matrices of E.R. with $d_N \underset{N\to\infty}{\longrightarrow} d > 0$ it is immediate to so that this property is not satisfied.

Proposition (M.11+)

If $\mathbf{A}_N$ and $\mathbf{B}_N$ are asymptotically traffic independent and $\mathbf{A}_N$ has the same limiting distribution of traffics as a unitary invariant family of random matrices, then $\mathbf{A}_N$ and $\mathbf{B}_N$ are asymptotically freely independent.

Example : $\mathbf{A}_N$ as the same limit as a family of independent GUE matrices.

Application for sparse graphs

Let G_N be a graph and A_N its adjacency matrix. For any $T = (V, E)$, $\mathrm{Tr}\, T^0(A_N)$ is the number of injective homomorphisms from T to G_N, that is the number of injective maps from the set of vertices of T into the one of G_N and from the set of edges of T into the one of G_N that preserve the adjacency and the orientation of the edges.

Proposition (M.11)

1. *If the degree of G_N is uniformly bounded, then A_N converges in distribution of traffics if and only if G_N converges in weak local topology.*
2. *Denote D_N the degree of a vertex uniformly chosen at random and assume $\mathbb{E}[{D_N}^k]$ converges for each k. If G_N converges in weak local topology then A_N converges in distribution of traffics, and its value is given by counting homomorphisms of T into the limit of (G, ρ) (chose arbitrary a root of T and restrict to root preserving homomorphism).*

Idea of the proof : fix an vertex r of T and restrict to vertex homomorphisms sending r to a vertex ρ_N of G_N uniformly chosen $\tau^0\big[(T,r)(A_N,\rho_N)\big] = \sum_{(H,s)\geq(T,r)} \tau^0\big[(T,r)(H,s)\big] \times \mathbb{P}\big((G_N,\rho_N)_p = (H,s)\big)$

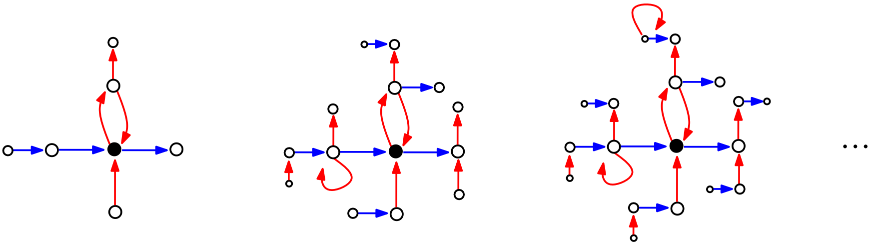
Corollary

Let $G_N^{(1)}, \dots, G_N^{(L)}$ be independent random graphs whose adjacency matrices are asymptotically traffic independent.

1. *If the limits of the $G_N^{(\ell)}$ are not regular graphs, then their adjacency matrices are not asymptotically freely independent.*
2. *[Accardi-Lenczewski-Salapata'07 or Cebron-Dahlqvist-M.'16] If the limits of the $G_N^{(\ell)}$ are deterministic graphs, then their adjacency matrices are asymptotically freely independent.*

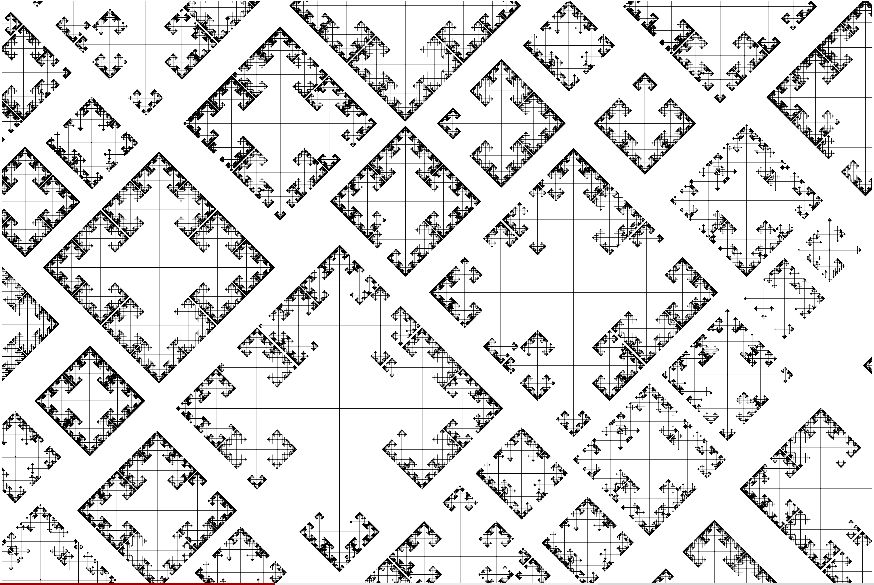
In particular, bond percolation on uniform regular graphs breaks asymptotic free independence of adjacency matrices.
It is not known if there exists infinite random regular graphs whose adjacency matrices are not freely independent.

Moreover, if two adjacency matrices of graphs A_N and B_N are asymptotically traffic free, the limit of (A_N, B_N) can be understood thanks to a "random free product" of the limiting graphs of A_N and B_N.



More precisely, the limits of quantities $\tau_{(A_N,B_N)}[T]$ are the number of injective homomorphisms of T into this graph that send a given vertex of T to the root of the random free product.

In particular, if the graphs are $(\mathbb{Z}, 0)$ with probability p, and the graph with a single vertex with no edges other, we obtain the following graphs :



Application for dense graphs [with Péché'14]

Our aim is to use this machinerie for the uniform simple regular with large degree d_N (possibly of order N). The difficulty is that we shall develop (a generalization of) the moment method which has not been considered in a "brutal" way [Tran-Vu-Wang'13].

Let G_N be a simple graph and A_N its adjacency matrix. Let multiply each non zero entries of A_N by i.i.d. random variables $(\xi_{i,j})_{i,j}$ with mean m_1 and second moment m_2. Let D_N be the degree of a vertex uniformly chosen, denote $d_N = \mathbb{E}[D_N]$ and $M_N = \frac{A_N - m_1 \frac{d_N}{N-1} J_N}{\sqrt{d_N(m_2 - {m_1}^2 \frac{d_N}{N-1})}}$.

We want to show that the limiting distribution of traffics of M_N is the same as for a GUE, in particular we must compute the limit of $\mathbb{E}\frac{1}{N}\mathrm{Tr}{M_N}^k$ for each k and understand the cancellations induced by the term $-m_1 \frac{d_N}{N} J_N$.

Assume first $m_1 = 0$ and $m_2 = 1$ and for simplicity $d_N \sim \alpha N$. Then for each $T = (V, E)$ we get

$$\tau_N^0\big[T(M_N)\big] = \tau_N^0\Big[T\Big(\frac{A_N}{m_2\sqrt{d_N}}\Big)\Big] = N^{V-1-\frac{|E|}{2}}\big(1+o(1)\big)\delta_N^0[T(A_N)]$$

where actually $\delta_N^0[T(A_N)] = \mathbb{E}\Big[\prod_{e\in T}\xi_e\Big]\mathbb{P}\big(T \subset G_N\big)$. Since the $\xi_{i,j}$ are centered, if there is an edge of multiplicity one, then $\tau_N^0\big[T(M_N)\big] = 0$. A classical relation between the number of edges and vertices in a connected graph yields

$$\tau_N^0\big[T(M_N)\big] = \mathbf{1}(T \text{ is a double tree}) {m_2}^{\frac{|E|}{2}}\mathbb{P}\big(T \subset G_N\big).$$

It turn out that when $\mathbb{P}\big(T \subset G_N\big) \sim \alpha^{\frac{|E|}{2}}$, then M_N as the same limiting distribution of traffics of a GUE matrix ! The convergence of $\mathbb{P}\big(T \subset G_N\big)$ is called convergence in topology of Graphons [Lovász-Szegedy'06].

The conclusion (generalizing the argument for $d_N \underset{N\to\infty}{\longrightarrow} \infty$ arbitrary) is that it is only reasonable to expect to get asymptotic free independence only on the condition that for any simple finite connected graph T with n edges, $\mathbb{P}\big(T \subset G_N\big) = \Big(\frac{d_N}{N}\Big)^n$

Theorem (M., Péché (14))

Let G_N be a random weighted graph invariant in law by relabeling of its vertices with mean degree d_N and i.i.d. weights. Given a finite simple graph T with edges $e_1, \dots, e_n$ denote $e_i(G_N) = \mathbf{1}(e_i \subset G_N)$. Assume that

$$\mathbb{E}\Big[\prod_{i=1}^{n}\big(e_i(G_N) - \frac{d_N}{N-1}\big)\Big] = \frac{d_N}{N-1} \times \varepsilon_N(T)$$

where $\varepsilon_N(T) = O(d_N^{-\frac{n}{2}})$ (in particular $\mathbb{P}\big(T \subset G_N\big) = \Big(\frac{d_N}{N}\Big)^n + O({d_N}^{-\frac{1}{2}})$). Then $\mathcal{M}_N$ has the same distribution of traffics as a GUE matrix and fulfills the assumptions of the asymptotic traffic independence theorem.

Theorem (M., Péché (14))

Assume $d_N, N - d_N \underset{N\to\infty}{\longrightarrow} \infty$ and there exists $\eta > 0$ such that $|\frac{N}{2} - d_N - \eta\sqrt{d_N}| \underset{N\to\infty}{\longrightarrow} \infty$. Then the above estimates holds for the d_N regular graph.

The proof is based on the switching method.

$$\mathbb{E}\Big[\prod_{i=1}^{n}\big(e_i(G_N) - \frac{d_N}{N}\big)\Big]$$

If $e_i \subset G_N$ then $e_i(G_N) - \frac{d_N}{N} = \frac{N-1-d_N}{N-1}$ and otherwise it equals $\frac{-d_N}{N-1}$, whose numerators correspond to a choice of an edge $f_i \not\subset G_N$ and $f_i \subset G_N$ respectively attached to e_i

Thank you for your attention !