On some problems and techniques in spectral geometry

Jean-Marc Bouclet
Institut de Mathématiques de Toulouse

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Outline of the talk

1. Background on Riemannian geometry
2. Examples of problems in spectral geometry
3. Basic strategy to handle these questions
4. Relationship with the geodesic flow
5. An idea of proof

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Rem: setting $\partial_x = (\partial_{x_1}, \ldots, \partial_{x_n})$,

$$\Delta_g = p(x, \partial_x) + \text{lower order terms (in } \partial_x)$$

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Fact: *If M is compact, there is an orthonormal basis of $L^2(M, dv_g)$ of eigenfunctions of L:*

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- Classical data: principal symbol, geodesic flow,...

in the semiclassical (or high energy) limit $\lambda_j \to \infty$.

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($h \sim$ effective Planck's constant), this is a special case of

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Question How small is $\epsilon(h)$?

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as long as $b - a \gg \tilde{\epsilon}(N_h)$ and with $\tilde{\epsilon}(N_h)$ either

$$O(N_h^{-\frac{1}{n}}) \qquad o(N_h^{-\frac{1}{n}}) \qquad O\left(\frac{N_h^{-\frac{1}{n}}}{\log N_h}\right)$$

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Open question: can one get similar improvment in non arithmetic cases ?

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Principle of the proof of Sogge's bounds : Choose ρ s.t. $\rho(0) = 1$ so that

$$\rho_{\epsilon(h)}(L_h - 1)e_h = e_h.$$

and suitably so that, with $\epsilon(h) = h$,

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with *suitable* $\rho \in \mathcal{S}(\mathbb{R})$
Principle of the proof of Sogge's bounds : Choose ρ s.t. $\rho(0) = 1$ so that

$$\rho_{\epsilon(h)}(L_h - 1)e_h = e_h.$$

and suitably so that, with $\epsilon(h) = h$,

$$||\rho_{\epsilon(h)}(L_h - 1)||_{2\to q} \lesssim h^{-\mu(q)}$$

Suitably means: $\hat{\rho}$ is compactly supported in some $[\delta, 2\delta]$, $0 < \delta \ll 1$

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For the Weyl law: Let $\chi \in C_0^\infty$ be equal to 1 on $[a, b]$. Then

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Rem 2: the control of $R(h)(=O(h^{-n}\epsilon(h)))$ is also based on this description

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One can then retrieve information on $\rho_{\epsilon(h)}(L_h - \lambda)\chi(L_h)$ by using stationary phase asymptotics in

$$h^{-d(D)} \int_{\mathbb{R}} \int_{\mathbb{R}^D} e^{\frac{\mathrm{i}}{h}\left(F(t,x,y,Z)-t\lambda\right)} A(t,x,y,Z,h)\hat{\rho}\left(\frac{t}{T(h)}\right) dZdt$$

or for the trace

$$h^{-1-d(D)} \int_M \int_{\mathbb{R}} \int_{\mathbb{R}^D} e^{\frac{\mathrm{i}}{h}\left(F(t,x,x,Z)-t\lambda\right)} A(t,x,x,Z,h)\hat{\rho}\left(\frac{t}{T(h)}\right) dZdtdv_g(x)$$

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Consequence: such approximations are in general limited to $|t| \leq c \log(1/h)$

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An idea of proof: the Bérard bound

Stationary phase in the integral

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Thank you for your attention

Thank you for your attention and happy new year

Some problems in spectral geometry

Some problems in spectral geometry

Let (M,g) be a Riemannian manifold of dimension n. Let dv_g be the natural measure and Δ_g be the Laplace-Beltrami operator, in local coordinates

$$dv_g = |g(x)| dx_1 \cdots dx_n, \qquad |g(x)| := \det\big(g_{jk}(x)\big)^{1/2} \tag{1}$$

$$\Delta_g = \sum_{j,k} g^{jk}(x)\partial_{x_j}\partial_{x_k} - \sum_{i,j,k} g^{jk}(x)\Gamma^i_{jk}(x)\partial_{x_i}$$

where $(g^{jk}(x)) = (g_{jk}(x))^{-1}$ and $g = \sum_{j,k} g_{jk}(x)dx_j dx_k$.

Proposition. *If M is closed (and connected), there exists an orthonormal basis $(\varphi_j)_{j\geq 0}$ of eigenfunctions of Δ_g*

$$-\Delta_g\varphi_j = \lambda_j\varphi_j, \qquad 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots, \qquad \lambda_j \uparrow +\infty.$$

Simple example: on $\mathbb{R}^n$

$$\chi(x) f(-h^2\Delta) v(x) = (2\pi h)^{-n} \int \int e^{\frac{i}{h}(x-y)\cdot\xi} \chi(x) f(|\xi|^2) v(y) dy d\xi$$

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Canonical example: gaussian wave packet

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$$(\mathcal{F}\psi_{z,\zeta})(\xi) = \pi^{-\frac{n}{4}} \exp\left(-\mathrm{i}z \cdot \xi - \frac{|\xi - \zeta|^2}{2}\right)$$

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$$(\mathcal{F}\psi_{z,\zeta})(\xi) = \pi^{-\frac{n}{4}} \exp\left(-iz \cdot \xi - \frac{|\xi - \zeta|^2}{2} \right)$$

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Main interests (for us) :

1. One can write "waves" (i.e. functions) as superposition of wave packets
2. The evolution of a wave packet under a Schrödinger flow can be described rather explicitly (in a suitable regime)

Wave packets: a review of basic facts

1. **Wave packet decomposition**

Wave packets: a review of basic facts

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Define the **Bargmann transform** of a function u by

$$Bu(z,\zeta) = \int_{\mathbb{R}^n} \overline{\psi_{z,\zeta}(x)}u(x)dx$$

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In other words

$$u(x) = (2\pi)^{-n}\int\int_{T^*\mathbb{R}^n}(Bu)(z,\zeta)\psi_{z,\zeta}(x)dzd\zeta$$

is a decomposition of u as a (continuous) sum of wave packets

2. **Evolution of wave packets under the Schrödinger equation**

Wave packets: a review of basic facts

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$$p_\nu(x,\xi) = \frac{|\xi|^2}{2} + \nu\frac{|x|^2}{2}, \qquad H_\nu = -\frac{\Delta}{2} + \nu\frac{|x|^2}{2}, \qquad \nu = 0, +1, -1$$

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Then

$$e^{-\mathrm{i}tH_\nu}\psi_{z,\zeta}(x) = \pi^{-\frac{n}{4}}\gamma_\nu^t \exp\mathrm{i}\left(S_\nu^t + \zeta_\nu^t\cdot(x-z_\nu^t) + \frac{\Gamma_\nu^t}{2}(x-z_\nu^t)\cdot(x-z_\nu^t)\right)$$

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and γ_ν^t, Γ_ν^t are given in term of the differential of flow $\Phi_{p_\nu}^t$,

$$D\Phi_{p_\nu}^t(z,\zeta) = \begin{pmatrix} A_\nu^t & B_\nu^t \\ C_\nu^t & D_\nu^t \end{pmatrix},$$

by

$$\Gamma_\nu^t = (C_\nu^t + \mathrm{i}D_\nu^t)(A_\nu^t + \mathrm{i}B_\nu^t)^{-1}, \qquad \gamma_\nu^t = \det(A_\nu^t + \mathrm{i}B_\nu^t)^{-1/2}.$$

Wave packets: a review of basic facts

Explicitly, we obtain

$$\begin{aligned}
\Gamma_0^t &= \frac{t+\mathrm{i}}{1+t^2} I_n, & \gamma_0^t = (1+\mathrm{i}t)^{-\frac{n}{2}} \\
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This allows in particular to read the profile and spreading of the packets:

$$
\begin{aligned}
|e^{\mathrm{i}tH_0}\psi_{z,\zeta}(x)| &= \frac{1}{(\pi(1+t^2))^{\frac{n}{4}}} \exp\left(-\frac{|x-z_0^t|^2}{2(1+t^2)}\right) \\
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Wave packets for semiclassical Schrödinger operators

From now on, we use a **semiclassical** normalization

$$\psi^h_{z,\zeta}(x) = (\pi h)^{-\frac{n}{4}} \exp\left(\frac{\mathrm{i}}{h}\zeta\cdot(x-z) - \frac{|x-z|^2}{2h}\right)$$

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Proposition [action of the symplectic group on the Siegel half space]
$A^t + \mathrm{i}B^t$ is invertible and

$$\Gamma^t := (C^t + \mathrm{i}D^t)(A^t + \mathrm{i}B^t)^{-1}$$

is symmetric complex, with positive definite imaginary part

Wave packets for semiclassical Schrödinger operators

Theorem (Hagedorn-Joye, Combescure-Robert) *In the limit $h \to 0$, and under general conditions on V,*

$$e^{-\mathrm{i}\frac{t}{h}H(h)}\psi^h_{z,\zeta}(x)$$

is well approximated by

$$(\pi h)^{-\frac{n}{4}}\gamma^t \mathcal{A}^h_t(x) \exp \frac{\mathrm{i}}{h}\left(S^t + \zeta^t \cdot (x - z^t) + \frac{\Gamma^t}{2}(x - z^t)\cdot(x - z^t)\right)$$

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Rem: on $\mathbb{R}^n$, $W_z^m = m - z$.

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Proposition. *Let U be a coordinate patch, with coordinates $y_1, \ldots, y_n$.*

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$$H(h) := -h^2 \frac{\Delta_g}{2} + V$$

$$(z^t, \zeta^t) = \Phi^t(z, \zeta), \qquad \text{Hamiltonian flow of } \frac{|\xi|_m^2}{2} + V(m)$$

Proposition. *Let U be a coordinate patch, with coordinates $y_1, \dots, y_n$. Along each trajectory starting at $(z, \zeta) \in T^*U$, one can define intrinsincally*

$$\Gamma^t : T_{z^t} M^{\mathbb{C}} \to T_{z^t} M^{\mathbb{C}}, \qquad \textit{where} \;\; T_{z^t} M^{\mathbb{C}} = T_{z^t} M \otimes \mathbb{C}$$

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$$\mathrm{Im} \langle \Gamma^t X, X \rangle_{z^t} > 0, \qquad X \neq 0, \; X \in T_{z^t} M$$

and satisfies the **Ricatti equation**

$$\nabla_{\dot{z}^t} \Gamma^t = -\mathrm{Hess}(V)_{z^t} - R_{z^t}(., \dot{z}^t)\dot{z}^t - (\Gamma^t)^2$$

where R_{z^t} is the Riemann tensor at z^t

Wave packets on Riemannian manifolds

Proof.
To construct Γ^t on $\mathbb{R}^n$, we have used the natural identifications

$$T_{(z,\zeta)}(T^*\mathbb{R}^n) = \mathbb{R}^n \oplus \mathbb{R}^n, \qquad T_{(z^t,\zeta^t)}(T^*\mathbb{R}^n) = \mathbb{R}^n \oplus \mathbb{R}^n$$

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   and split along **horizontal** and **vertical** spaces

$$T_{(z^t,\dot{z}^t)}(\mathcal{I}_g T^*M) = \mathcal{H}_{(z^t,\dot{z}^t)} \oplus \mathcal{V}_{(z^t,\dot{z}^t)}$$

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This gives a natural block decomposition

$$d\left(\mathcal{I}_g \circ \Phi^t\right) = \begin{pmatrix} \mathcal{L}_A & \mathcal{L}_B \\ \mathcal{L}_C & \mathcal{L}_D \end{pmatrix} : \mathbb{R}^n_y \oplus \mathbb{R}^n_\eta \to \mathcal{H}_{(z^t,\dot{z}^t)} \oplus \mathcal{V}_{(z^t,\dot{z}^t)}$$

Wave packets on Riemannian manifolds

Proof (continued). One *can* then define

$$\big(\mathcal{L}_C + \mathrm{i}\mathcal{L}_D\big)\big(\mathcal{L}_A + \mathrm{i}\mathcal{L}_B\big)^{-1} : \mathcal{H}^{\mathbb{C}}_{(z^t,\dot{z}^t)} \to \mathcal{V}^{\mathbb{C}}_{(z^t,\dot{z}^t)}$$

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Wave packets on Riemannian manifolds

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Rem. If $(\tilde{y}_1, \dots, \tilde{y}_n)$ are other coordinates on U, the matrix of Γ^t is changed into

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Wave packets on Riemannian manifolds

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Wave packets on Riemannian manifolds

Definition of gaussian wave packets

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Definition of gaussian wave packets *Let* $\rho \in C_0^\infty(-r_0, r_0)$, *equal to* 1 *near* 0.

Wave packets on Riemannian manifolds

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$$\Psi^h_{z,\zeta}(m) := (\pi h)^{-\frac{n}{4}} \gamma^0 \exp \frac{\mathrm{i}}{h} \left(\zeta \cdot W_z^m + \frac{1}{2} \langle \Gamma^0 W_z^m, W_z^m \rangle_z \right) \rho\left(d_g(z, m)\right),$$

for $m \in M$ and $(z, \zeta) \in T^*U$ (i.e. $\zeta \in T_z^* U$)

$$\gamma^0 = det\big(g_{jk}(y(z))\big)^{-\frac{1}{4}}$$

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$$(2\pi h)^{-n} \int\int_{T^*U} B_h u(z,\zeta) \Psi^h_{z,\zeta} dz d\zeta = a(h) u$$

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which turns out to be equivalent to

$$\frac{d}{dt}\left(T[E_t\cdot,\dots,E_t\cdot]\right) = F[E_t\cdot,\dots,E_t\cdot]$$

with $E_t := d\pi(\mathcal{L}_A + \mathrm{i}\mathcal{L}_B) : \mathbb{C}^n \to T_{z^t}M \otimes \mathbb{C}$ ($d\pi =$ projection from the horizontal space at $(z^t, \dot{z}^t)$ to the tangent space at z^t)

$\Longrightarrow$ Control on the exponential growth in time of $T_j(t, z^t, \zeta^t, .)$.

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$$K_t^h(m, m') = h^{-\frac{3n}{2}} \int\int_{T^*U} b_h(t, z, \zeta, m, m') \exp \frac{\mathrm{i}}{h} F(t, z, \zeta, m, m') dz d\zeta$$

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Proof:

$$e^{-\mathrm{i}\frac{t}{h}H(h)}A_h u = (2\pi h)^{-n}\int\int_{T^*U} e^{-\mathrm{i}\frac{t}{h}H(h)}\Psi^h_{z,\zeta}\left\langle A_h^* a_h^{-1}\Psi^h_{z,\zeta}, u\right\rangle_{L^2(M)} dzd\zeta$$

Thank you for your attention