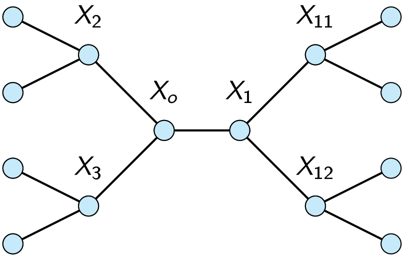
Spectral measures of factor of i.i.d. processes on the regular tree

Ágnes Backhausz
Eötvös Loránd University and Rényi Institute, Budapest
Joint work with Bálint Virág

7 January 2016, CIRM, Luminy

Example: Gauss Markov process

An invariant random process on the infinite 3-regular tree (for $0 < s < 1$):



$$X_1 = sX_o + \sqrt{1-s^2}\varepsilon_1; \qquad X_2 = sX_o + \sqrt{1-s^2}\varepsilon_2;$$
$$X_3 = sX_o + \sqrt{1-s^2}\varepsilon_3; \qquad X_{11} = sX_1 + \sqrt{1-s^2}\varepsilon_{11},$$

where $X_o, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_{11}, \varepsilon_{12}, \ldots$ are independent $N(0,1)$ random variables.

Factor of i.i.d. processes

- $T_d = (V, E)$ is the rooted d-regular tree ($d \geq 3$) with root o.
- Let $\Omega = \mathbb{R}^V$, and $\mathbb{P}$ be the product measure of $N(0, 1)$ distributions.
- Fix an $f \in L^2(\Omega, \mathbb{P})$, which is invariant under the root-preserving automorphisms of T_d.

Factor of i.i.d. processes

- $T_d = (V, E)$ is the rooted d-regular tree ($d \geq 3$) with root o.
- Let $\Omega = \mathbb{R}^V$, and $\mathbb{P}$ be the product measure of $N(0, 1)$ distributions.
- Fix an $f \in L^2(\Omega, \mathbb{P})$, which is invariant under the root-preserving automorphisms of T_d.
- For every $v \in V$ and $\omega \in \Omega$, let X_v be the value of f on ω with the root placed to $v \Rightarrow (X_v)_{v \in V}$ is a **factor of i.i.d. process**.

Factor of i.i.d. processes

- $T_d = (V, E)$ is the rooted d-regular tree ($d \geq 3$) with root o.
- Let $\Omega = \mathbb{R}^V$, and $\mathbb{P}$ be the product measure of $N(0, 1)$ distributions.
- Fix an $f \in L^2(\Omega, \mathbb{P})$, which is invariant under the root-preserving automorphisms of T_d.
- For every $v \in V$ and $\omega \in \Omega$, let X_v be the value of f on ω with the root placed to $v \Rightarrow (X_v)_{v \in V}$ is a **factor of i.i.d. process**.
- **Linear factor of i.i.d.** Let $\alpha \in \ell^2(V)$, and $f(\omega) = \sum_{v \in V} \alpha_v \omega_v$.

Factor of i.i.d. processes

- $T_d = (V, E)$ is the rooted d-regular tree ($d \geq 3$) with root o.
- Let $\Omega = \mathbb{R}^V$, and $\mathbb{P}$ be the product measure of $N(0, 1)$ distributions.
- Fix an $f \in L^2(\Omega, \mathbb{P})$, which is invariant under the root-preserving automorphisms of T_d.
- For every $v \in V$ and $\omega \in \Omega$, let X_v be the value of f on ω with the root placed to $v \Rightarrow (X_v)_{v \in V}$ is a **factor of i.i.d. process**.
- **Linear factor of i.i.d.** Let $\alpha \in \ell^2(V)$, and $f(\omega) = \sum_{v \in V} \alpha_v \omega_v$.
- Question. For which s is the Gauss Markov process a factor of i.i.d. process?

Factor of i.i.d. constructions

- large independent sets [Csóka–Gerencsér–Harangi–Virág 2013, Hoppen–Wormald 2014] in random 3-regular graphs
- perfect mathcings [Lyons–Nazarov 2011, Csóka–Lippner 2012+] on non-amenable Cayley graphs
- 4-regular spanning forests [Gaboriau–Lyons 2009, Kun 2013+] on non-amenable Cayley graphs

Covariance structure

Let $c_X : V(T_d) \to \mathbb{R}$ be the **covariance structure** of an invariant process $(X_v)_{v \in V}$ defined by

$$c_X(v) = \mathrm{cov}(X_o, X_v) \qquad (v \in V(T_d)).$$

Covariance structure

Let $c_X : V(T_d) \to \mathbb{R}$ be the **covariance structure** of an invariant process $(X_v)_{v \in V}$ defined by

$$c_X(v) = \mathrm{cov}(X_o, X_v) \qquad (v \in V(T_d)).$$

The covariance structure of the Gauss Markov process $(X_v)_{v \in V}$ is given by

$$\mathrm{cov}(X_o, X_v) = s^k \text{ if } \mathrm{dist}(o, v) = k.$$

Spectral measure

Let $\alpha : V(T_d) \to \mathbb{R}$ be an arbitrary function. The adjacency operator A is defined by

$$[A\alpha](v) = \sum_{w \sim v} \alpha(w) \qquad (v \in V(T_d)).$$

Spectral measure

Let $\alpha : V(T_d) \to \mathbb{R}$ be an arbitrary function. The adjacency operator A is defined by

$$[A\alpha](v) = \sum_{w \sim v} \alpha(w) \qquad (v \in V(T_d)).$$

We can choose a finite measure μ_X on $[-d, d]$ such that

$$\langle A^k \delta_o, c_X \rangle = \int_{-d}^{d} t^k \, d\mu_X(t) \qquad (k \geq 0).$$

This μ_X is the **spectral measure** of the process X.

Spectral measure

Let $\alpha : V(T_d) \to \mathbb{R}$ be an arbitrary function. The adjacency operator A is defined by

$$[A\alpha](v) = \sum_{w \sim v} \alpha(w) \qquad (v \in V(T_d)).$$

We can choose a finite measure μ_X on $[-d, d]$ such that

$$\langle A^k \delta_o, c_X \rangle = \int_{-d}^{d} t^k \, d\mu_X(t) \qquad (k \geq 0).$$

This μ_X is the **spectral measure** of the process X.

The spectral measure of the i.i.d. process is the Kesten–McKay measure ν, which has density

$$h(t) = \begin{cases} \frac{d}{2\pi} \frac{\sqrt{4(d-1)-t^2}}{d^2-t^2} & t \in \left[-2\sqrt{d-1}, 2\sqrt{d-1}\right]; \\ h(t) = 0 & \text{otherwise} \end{cases}$$

with respect to the Lebesgue measure.

Examples for spectral measure

1. **Gaussian wave functions** corresponding to $\lambda \in [-d, d]$: an invariant random process $(X_v)_{v \in V}$ satisfying the following with probability 1:

$$\lambda X_v = \sum_{w \sim v} X_w \qquad (v \in V(T_d)).$$

Examples for spectral measure

1. **Gaussian wave functions** corresponding to $\lambda \in [-d, d]$: an invariant random process $(X_v)_{v \in V}$ satisfying the following with probability 1:

$$\lambda X_v = \sum_{w \sim v} X_w \qquad (v \in V(T_d)).$$

   This process exists for $\lambda \in [-d, d]$ (Harangi–Virág, 2015).

   The spectral measure μ_X is an atomic measure at λ.

Examples for spectral measure

1. **Gaussian wave functions** corresponding to $\lambda \in [-d, d]$: an invariant random process $(X_v)_{v \in V}$ satisfying the following with probability 1:

$$\lambda X_v = \sum_{w \sim v} X_w \qquad (v \in V(T_d)).$$

   This process exists for $\lambda \in [-d, d]$ (Harangi–Virág, 2015).

   The spectral measure μ_X is an atomic measure at λ.

2. **Linear block factors**: Y is the i.i.d. process, $X = p(A)Y$. Then the spectral measure μ_X has density p^2 with respect to $\mu_Y = \nu$.

Examples for spectral measure

1. **Gaussian wave functions** corresponding to $\lambda \in [-d, d]$: an invariant random process $(X_v)_{v \in V}$ satisfying the following with probability 1:

$$\lambda X_v = \sum_{w \sim v} X_w \qquad (v \in V(T_d)).$$

   This process exists for $\lambda \in [-d, d]$ (Harangi–Virág, 2015).

   The spectral measure μ_X is an atomic measure at λ.

2. **Linear block factors**: Y is the i.i.d. process, $X = p(A)Y$. Then the spectral measure μ_X has density p^2 with respect to $\mu_Y = \nu$.

Linear factor of i.i.d. processes: this can be extended to $\ell^2(V)$ functions.

Spectral measures of factor of i.i.d. processes

Let ν be the Kesten–McKay measure (the spectral measure of i.i.d.).

Theorem (B–Virág, 2015+)

The following are equivalent for a finite measure μ on $[-d, d]$.

1. $\mu = \mu_X$ *for some linear factor of i.i.d. process* X.
2. $\mu = \mu_X$ *for some factor of i.i.d. process* X.
3. μ *is absolutely continuous with respect to* ν.

Spectral measures of factor of i.i.d. processes

Let ν be the Kesten–McKay measure (the spectral measure of i.i.d.).

Theorem (B–Virág, 2015+)

The following are equivalent for a finite measure μ on $[-d, d]$.

1. $\mu = \mu_X$ *for some linear factor of i.i.d. process* X.
2. $\mu = \mu_X$ *for some factor of i.i.d. process* X.
3. μ *is absolutely continuous with respect to* ν.

Theorem (B–Virág, 2015+)

The following are equivalent for a finite measure μ on $[-d, d]$.

1. $\mu = \mu_X$ *for some weak limit of linear factor of i.i.d. processes* X.
2. $\mu = \mu_X$ *for some weak limit of factor of i.i.d. processes* X.
3. $\mathrm{supp}(\mu) \subseteq \mathrm{supp}(\nu)$.

$\overline{d}_2$**-distance** of invariant random processes with marginals having finite second moments:

$$\overline{d_2}^2(X, Z) = \inf \left\{ \mathbb{E}((X'_o - Z'_o)^2) \right\},$$

where the infimum is taken over all invariant couplings (X', Z') of X and Z.

$\overline{d_2}$ and spectral measure

$\overline{d}_2$**-distance** of invariant random processes with marginals having finite second moments:

$$\overline{d_2}^2(X,Z) = \inf\left\{\mathbb{E}((X'_o - Z'_o)^2)\right\},$$

where the infimum is taken over all invariant couplings (X', Z') of X and Z.

Proposition

Let $(X_v)_{v\in V}$ and $(Z_v)_{v\in V}$ be invariant random processes on T_d with marginals having mean zero and finite second moments. Then we have

$$\overline{d_2}^2(X,Z) \geq \mathbb{E}(X_o^2) + \mathbb{E}(Z_o^2) - \sqrt{\left(\mathbb{E}(X_o^2) + \mathbb{E}(Z_o^2)\right)^2 - 4d_{TV}(\mu_X, \mu_Z)}$$

Applications

- A Gauss Markov process is factor of i.i.d. if and only if $s \leq \frac{1}{\sqrt{d-1}}$.

Applications

- A Gauss Markov process is factor of i.i.d. if and only if $s \leq \frac{1}{\sqrt{d-1}}$.

Applications

- A Gauss Markov process is factor of i.i.d. if and only if $s \leq \frac{1}{\sqrt{d-1}}$.
- For $\lambda \in [-2\sqrt{d-1}, 2\sqrt{d-1}]$, the Gaussian wave function is limit of factor of i.i.d. in distribution, but its $\overline{d}_2^2$-distance is $\sqrt{2}$ from every factor of i.i.d. process.
- For $\lambda \notin [-2\sqrt{d-1}, 2\sqrt{d-1}]$, the Gaussian wave function is not a limit of factor of i.i.d. processes.

Applications

- A Gauss Markov process is factor of i.i.d. if and only if $s \leq \frac{1}{\sqrt{d-1}}$.
- For $\lambda \in [-2\sqrt{d-1}, 2\sqrt{d-1}]$, the Gaussian wave function is limit of factor of i.i.d. in distribution, but its $\overline{d}_2^2$-distance is $\sqrt{2}$ from every factor of i.i.d. process.
- For $\lambda \notin [-2\sqrt{d-1}, 2\sqrt{d-1}]$, the Gaussian wave function is not a limit of factor of i.i.d. processes.
- Gaussian wave functions are orthogonal in every coupling:

$$\mathbb{E}(X_o Z_o) = 0 \text{ and } \overline{d}_2^2(X, Z) = \sqrt{2},$$

  holds if X and Z are Gaussian wave functions corresponding to different eigenvalues with marginals having mean 0 and variance 1.

Extensions

- Similar results hold for vertex-transitive graphs.
- Process spectrum of a graph G:

$$\mathrm{psp(G)} = \overline{\bigcup_X \mathrm{supp}(\mu_X)},$$

where the union is for all invariant processes X on G having marginals with finite second moments.

Extensions

- Similar results hold for vertex-transitive graphs.
- Process spectrum of a graph G:

$$\mathrm{psp}(\mathrm{G}) = \overline{\bigcup_X \mathrm{supp}(\mu_X)},$$

  where the union is for all invariant processes X on G having marginals with finite second moments.
- Process spectral radius: $\varrho^+(G) = \sup\{x < d : x \in \mathrm{psp}(G)\}$.
- **Open question:** is there a graph G for which the spectral radius, the process spectral radius and d are all different?

Open questions

1. Is there a graph G for which the spectral radius, the process spectral radius and d are all different?
2. Is the family of factor of i.i.d. processes closed under $\bar{d}_2$-convergence?
3. For which parameters is the Ising model (on the regular tree) a factor of i.i.d. process?