Geometry of univariate stability: continuity argument, symmetric products and stability theories

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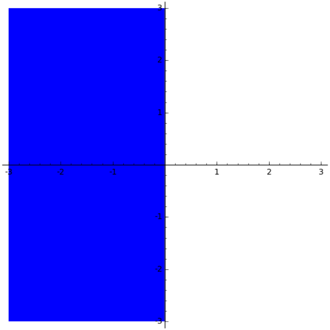
Stability of polynomials: Hurwitz stability

Polynomial $p(x)$ is Hurwitz stable if all of its roots are located in the left half-plane.

$$\forall x \in \mathbb{C}(p(x) = 0) \Rightarrow Re(x) < 0$$



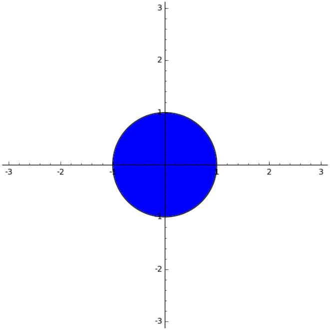
Stability of matrices: Hurwitz stability

Matrix A is Hurwitz stable iff all of its eigenvalues(roots of characteristic polynomial) are located in the left half-plane.
This is equivalent to the statement that all solutions of system of differential equations
$\frac{dx}{dt} = Ax(t)$ tends to 0 as $t \to \infty$.

Stability of polynomials: Schur stability

Polynomial $p(x)$ is Schur stable if all of its roots are located at the unit circle.

$$\forall x \in \mathbb{C}(p(x) = 0) \Rightarrow |x| < 1$$



Stability of matrices: Schur stability

Matrix A is stable if all of its eigenvalues are located at the unit circle.
This is equivalent to the statement that all solutions of system of difference equations

$$x(t+1) = Ax(t)$$

tends to 0 as $t \to \infty$

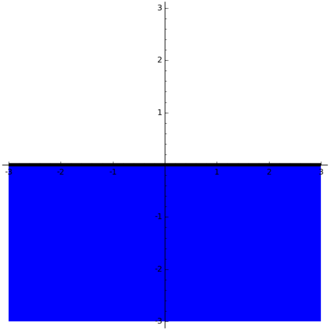
Hyperbolicity as stability.

Real coefficients

Polynomial with real coefficients is called hyperbolic if it has only real roots.

Complex coefficients

Let us call a polynomial with complex coefficients quasihyperbolic if all of its roots are located in open down half-plane.
Then hyperbolicity becomes *border of* quasihyperbolicity.



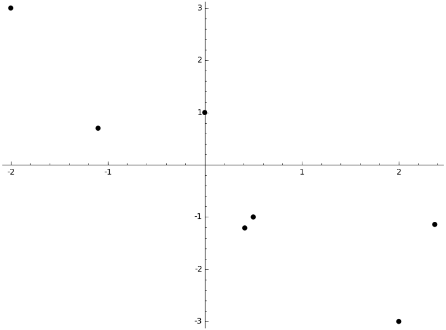
Pole placement problems

Polynomials

Set of stable points on a complex plane is a fixed finite set S..

Control-theoretic source

Let (A, B, C) be linear control system defined by a triple of matrices of sizes $n \times n$, $m \times n$, $n \times p$ respectively.
Then the problem of finding matrix K such that all eigenvalues of $A + BKC$ lies in S is an output feedback pole placement problem.



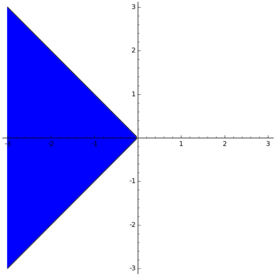
Superstabilisability

Superstability (modification of Polyak B.T,, Shcherbakov P.S. 2002.)

Superstability is a condition when ∞-norm of all solutions of system of linear differential (difference) equations *monotonically* tends to 0 as $t \to \infty$.

Hurwitz superstabilisability

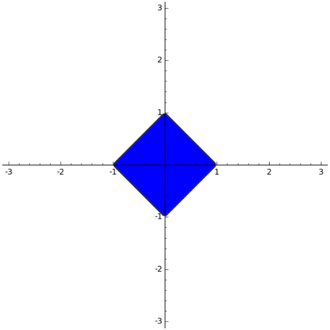
If all eigenvalues of matrix A are located in $\{z \mid |Im\, z| - Re\, z > 0\}$ then system $\frac{dx}{dt} = Ax(t)$ is superstable.



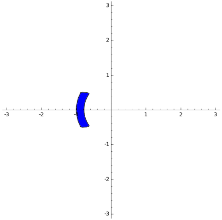
Schur superstabilisability

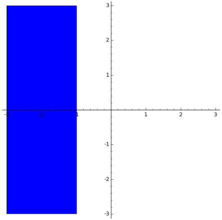
Schur superstabilisability

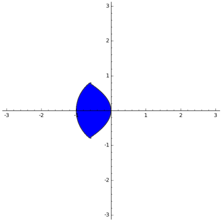
If all eigenvalues of matrix A are located in $\{z||Re\, z| + |Im\, z| < 1\}$ then system $x(t+1) = Ax(t)$ is superstable.

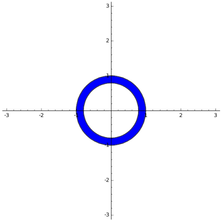


Other simplest natural control-theoretic regions









General stability theories: Root clustering. Computational methods.

- R.E. Kalman, 1969: General problem statement
- Boolean combinations of circles and half-planes(since 1977)
- Regions “transformable” to classical (S. Gutman, E. Jury, B. Barmish et al. since 1981)
- Regions defined by LMI(M. Chilali, P. Gahinet, D.Henrion et. al. since 1996)
- (Elementary) semialgebraic sets (J.-B. Lasserre, 2004)
- Cassini ovals(V.G. Melnikov, 2011)
- Instability regions with many connected components (V.G. Melnikov, N.A. Dudarenko, 2014)

Geometry of classical stabilities: continuity argument and beyond

- Root-coefficient correspondence. F. Viete, A. Girard, XVI-XVII centuries.
- Hurwitz stability for polynomials: I. Vyshnegradski(1876).
- Geometry of robust stability problems. D-decomposition. A.A. Andronov, Yu.I. Neimark et al. (since 1940s).
- Singularities of stability borders and topology of "complements to discriminants". V.I.Arnold school since 1970s.
- Stability and applied singularity theory (A.A. Maylibayev, A.P. Seyranian 1990-s–2000s).
- Continuity-based proof of Routh-Hurwitz criterium. (G. Meinsma, 1994).
- Algebro-geometric methods for D-decomposition (B.T Polyak, E.Gryazina (2004-2008),author(as O.O. Vasilév, since 2012)
- Symmetric products and topology of the space of Hurwitz and Schur polynomials (B. Aguirre-Hernandez, J.L. Cisneros-Molina, M.E. Frias-Armenta, 2012)

(In)stability regions in parameter space: *D*-decomposition

D-decomposition

Consider a family of polynomial or matrices affinelyparametrised by finite vector of real parameters $k = (k_1, \ldots, k_l)$.
D-decomposition is a partition of parameter space $\mathbb{R}^l$ into regions with same number of stable roots.

Geometry of PI and PID-controllers

Definition

PID-controller is a 3-parametric affine family of polynomials $IR(s) + s(Q(s) + PR(s)) + Ds^2R(s)$, $deg\ Q(s) > deg\ R(s)$.
PI-controller is a 2-parametric affine family of polynomials, given by PID-controller with $D = 0$.
Most of the industrial controllers are of this type.

Non-connectedness

It is known that stability region could be non-connected and non-convex.
If P is fixed, the stability region is union of finite number of convex polygons(Ho, Datta, Bhattacharya, 1998).

Main idea of an approach/Meta-program

- Stability theory is a stratification of set possible values of a root.
- There exist (infinite-dimensional) universal spaces of stability problems. That spaces are symmetric products of stability theories.
- Individual stability problem – is an affine section of that universal space. Stability regions – are affine sections of universal stability region.
- There exist a moduli space of stability problems of give dimension. It is a quotient of Grassmann variety. It is a filtered stratified space.
- For concrete computations(i.e. optimization) on stability problems and their spaces one should find some good embedding of that space into real space, and make computations on it.

Filtered spaces: definitions

Definition

Filtered real algebraic variety L a infinite sequence of closed embeddings of real algebraic varieties $L_0 \rightarrow^{\lambda_0} L_1 \rightarrow^{\lambda_1} \dots$.
Morphism between filtered real algebraic varieties $\varphi L \rightarrow R$, is a sequence of morphisms $\varphi_i \colon L_i \rightarrow R_i$ that commutes with embeddings.

Note

In general one may need to consider an objects like "real closed ind-schemes" whatever it should mean.

Filtered action

Let $G_0 \subseteq G_1 \subseteq \ldots = G$ be a filtered group and L be a filtered real variety.
Define *a filtered action of G on L* as a sequence of action G_i on L_i that commutes with embeddings.

Notations

- $U_0 \subset U_1 \subset \ldots \subset U_i \subset$ a filtration of spaces of all polynomials with complex coefficients. Here U_i is a $(2i+2)$-dimensional space of polynomials degree less than i.
- $\mathbb{C}^\infty$ - filtered space of complex sequences with finite number of non-zero elements;
- $Mat(\mathbb{C}, \infty)$ - filtered space of square matrices with finite number of non-zero entries;
- $Gl(\mathbb{C}, \infty)$ - group of invertible transformations of $\mathbb{C}^\infty$;
- Σ^∞ - infinite symmetric group (permutations with finite number of non-stable points);

Symmetric product

Definition

Infinite symmetric product of real algebraic variety R with marked point e is a filtered real algebraic variety $R^{(\infty)}$ given as sequence of quotients defined by filtered action of filtered group $\Sigma^\infty = \Sigma_1 \subset \Sigma_2 \subset \dots$ on filtered space $R \to^{\varphi_1} R^2 \to \dots,$
$\varphi_i \colon (r_1, \dots, r_i) \mapsto (r_1, \dots r_i, e)$.

Note

In general, symmetric products of real algebraic varieties could be not real algebraic varieties, but only semialgebraic spaces(abstract semialgebraic sets).
Infinite symmetric products of $\mathbb{C}$ and of $\mathbb{C}\mathbf{P}^1$ are filtered real algebraic varieties.

Stability theory: definition

- Stability theory is a triple $S = (\mathbb{C}P^1, \Omega, \infty)$.
- $\mathbb{C}P^1$ here considered as a real algebraic variety and Ω is a semi-algebraic subset of it. Let us fix an affine map of $\mathbb{C} = \mathbb{C}P^1 \setminus \{\infty\}$.
- $\mathbb{C}P^1$ as an underlying space of stability theory admits a canonical stratification $Str(S)$ into sets $\Omega = \Omega_s$, $\overline{\Omega} \setminus \Omega = \Omega_{ss}$, $\mathbb{C}P^1 \setminus \overline{\Omega} = \Omega_{un}$, where closure is euclidean.

Stable, semistable and unstable roots

Roots

Let $p \in \mathbb{R}[i][x]$ be a polynomial. Call root r of p Ω-stable if $p \in \Omega$, call it Ω-semistable if $p \in \overline{\Omega} \setminus \Omega$, where closure is considered to be euclidean. Otherwise call it Ω-unstable.
Each polynomial p has it's own Ω-*stability* index defined as triple (r_s, r_{ss}, r_{un}), $r_s + r_{ss} + r_{un} = \deg p$.

D-stratification

Define a D-stratification D^n_S of U^n as a most rude stratification of U^n into connected regions with the same stability index relative to stability theory $S = (\mathbb{C}\mathbf{P}^1, \Omega, \infty)$.
Denote by $(k, l, m)_\Omega$ a union of strata with stability index (k, l, m).

Embeddings of strata

Let $\infty \in \Omega_i, \quad i \in \{s, ss, un\}$. Let $k = (k_s, k_{ss}, k_{un})_\Omega$
$l = (l_s, l_{ss}, l_{un})_\Omega k_i \leq l_i$ be strata of D_S.
Then they are either mutually disjoint or $k \subseteq l$. Latter case is true iff for each $j \in \{s, ss, un\} \setminus \{i\}$ $k_j = l_j$.

Root-coefficient correspondence and symmetric product morphism

Fix a stability theory S with stability set Ω.
Let $(\mathbb{C}\mathbf{P}^1)^\infty$ be a filtered space of finite sequences of points from $\mathbb{C}\mathbf{P}^1$ with a natural embeddings $(\mathbb{C}\mathbf{P}^1)^n \to (\mathbb{C}\mathbf{P}^1)^{n+1}, s \mapsto (s, \infty)$, and stratification induced by stability theory S.
All morphisms in the following diagram below are morphisms of filtrations of stratified real algebraic varieties

$$(\mathbb{C}\mathbf{P}^1)^\infty \twoheadrightarrow^{\eta\Sigma^\infty} (\mathbb{C}\mathbf{P}^1)^{(\infty)} \xrightarrow{\sim} \mathbb{C}\mathbf{P}^\infty = \mathbf{P}(U_\Omega) \twoheadleftarrow U_\Omega \setminus \{0\} \hookrightarrow U_\Omega$$

Matrix-polynomial duality

Let us denote by $U_{\frac{1}{\Omega}}$ space of polynomials stratified by a stability theory $(\mathbb{C}\mathbf{P}^1, \{\frac{1}{\lambda}, \lambda \in \Omega\}, \infty)$. Consider $Mat(\mathbb{C}, \infty)$ as a space stratified by the same stability theory.
Then following diagram is commutative in category of filtrations of stratified real varieties:

$$\begin{tikzcd}
 & (\mathbb{C}\mathbf{P}^1)^\infty \arrow[r, "\pi_{\Sigma^\infty}"] & (\mathbb{C}\mathbf{P}^1)^{(\infty)} \cong \mathbb{C}\mathbf{P}^\infty & U_\Omega \setminus \{0\} \arrow[l, "\pi_{\mathbb{C}^*}"'] \arrow[r] & U_\Omega \\
\mathbb{C}^\infty \arrow[ur, "\{\frac{1}{x}\}^\infty"] \arrow[dr, "diag"'] & & \mathbb{C}\mathbf{P}^\infty = \mathbf{P}(U_{\frac{1}{\Omega}}) \arrow[u] & & \\
 & Mat(\mathbb{C}, \infty) \arrow[r, "\pi_{Gl(\mathbb{C},\infty)}"] & U_{\frac{1}{\Omega}} \setminus \{0\} \arrow[u, "\pi_{\mathbb{C}^*}"] & U_{\frac{1}{\Omega}} \arrow[l] &
\end{tikzcd}$$

Deformation equivalence

Matrix-polynomial duality is equivalent to duality between polynomial deformations:

$$\begin{array}{ll} \text{Polynomials:} & a_n z^n + \ldots + a_0 \mapsto \epsilon z^{n+1} + a_n z^n + \ldots + a_0 \\ \text{Matrices:} & a_n z^n + \ldots + a_0 \mapsto z(a_n z^n + \ldots + a_0) + \epsilon. \end{array}$$

Structure of strata: connected components

Let $(k, l, m) \in \mathbb{N}^3$. Then there are

$$C^{k}_{|\pi_0(\Omega_s)|+k-1}\, C^{l}_{|\pi_0(\Omega_{ss})|+l-1}\, C^{m}_{|\pi_0(\Omega_{un})|+m-1}$$

stratas of U_Ω with stability index (k, l, m).

Symmetric product of graph

Let G be a graph with marked vertice e.
n-th symmetric product of G $G^{(n)}$ is a quotient G^n by natural action of symmetric group Σ_n.
An infinite symmetric product of G is a sequence of embeddings of $G^{(n)}$, produced by adding $\{e\}$ to each multisubset of $V(G)$ forming vertice of $G^{(n)}$.
We denote it as denoted as $G^{(\infty)}$.

Adjacency on topological spaces

Assume that

- $L = \{L_1, \ldots, L_k\}$ is a decomposition of topological space T with marked point $e \in F \in L$ into finite number of mutually disjoint subsets.
- $T^{(\infty)}$ is an infinite symmetric product of (T, e).
- $L^{(\infty)}$ is a decomposition of $T^{(\infty)}$ induced by L.

Then infinite symmetric product of adjacency graphs with marked point G_L^T is an adjacency graphs of infinite symmetric product of decomposition $(G_L^T, F)^{(\infty)} \cong G_{L^\infty}^{(T,e)^\infty}$.

Adjacency for stability theories: general case

Assume that Ω is either closed or $\overline{\Omega} \setminus \Omega \not\subset int(\overline{\Omega})$.
Take

$$k = (k_s, k_{ss}, k_{un}), l = (l_s, l_{ss}, l_{un}) \in \mathbb{N}^3, \sum_i k_i = n, \sum_i l_i = m; m \leq n.$$

Then $\overline{P(k_\Omega)} \cap \overline{P(l_\Omega)} \neq \varnothing$ iff either $m = n$ or for each i such that $l_i < k_i$ corresponding strata of stability theory Ω_i is an unbounded region of $\mathbb{C}$.
In particular, Hurwitz and Schur stability theories give rise to non-isomorphic stratifications.

Adjacency for stability theories: degenerate case

Let $k = (k_s, k_{ss}, k_{un}), l = (l_s, l_{ss}, l_{un}) \in \mathbb{N}^3, \sum_i k_i = n, \sum_i l_i = m$; $m \leq n$. Then $\overline{P(k_\Omega)} \cap \overline{P(l_\Omega)} \neq \varnothing$ iff one of the following assumptions holds

1. $m = n$, $k_{ss} = l_{ss}$,
2. $m = n$. $k_{un} = l_{un}$,
3. $m < n$, Ω_s is unbounded, Ω_{ss} is bounded, Ω_{un} is unbounded, $k_{ss} = l_{ss}$,
4. $m < n$, Ω_s is unbounded, Ω_{ss} is unbounded, Ω_{un} is bounded, $k_{un} = l_{un}$,
5. $m < n$, Ω_s is unbounded, Ω_{ss} is bounded, Ω_{un} is bounded, $k_s < l_s$,
6. $m < n$, Ω_s is bounded, Ω_{ss} is bounded, Ω_{un} is unbounded, $k_{un} < l_{un}$.

Topology of strata. Fundamental group.

Fundamental group

Let Ω_{ss} be a irreducible smooth real algebraic curve. Then for each $(k, l, m)_\Omega$ and $x \in (k, l, m)_\Omega$ $\pi_1((k, l, m)_\Omega, x)$ is product of free free groups.

Scheme of the proof

- Note that all strata are homeomorphic to products of symmetric products of Ω_i
- Note that Ω_i are homotopically equivalent to bouquets of circles.
- Apply B.W.Ong(2003) theorem on homotopical type of symmetric product of bouquets of circles (Ong's proof is based on pole placement example!)
- Compute fundamental group using results of A. Hattori(1975).

Topology of strata: demixing components

- Let $(k, l, m)_\Omega$ be a strata.
- It is homeomorphic to $\Omega_s^{(k)} \times \Omega_{ss}^{(l)} \times \Omega_{un}^{(m)}$.
- In particular, in case of Hurwitz and Schur stability theories stratas $(k, 0, m)$ are contractible and strata $(k, l, m), l > 0$ are homotopically equivalent to S^1 (H.R. Morton, 1967).
- For the pole placement problem strata of type $(0, 0, m)$ are complements to configurations of hyperplanes.

Topology of strata. "Bones" of torus

Let $T^n = (S^1)^n$ be an n-dimensional torus. Denote by T^n_q a union of all q-dimensional coordinate subtorii

$$\cup_{I \subseteq \{1,\dots,n\}, |I|=q} \{(s_1, \dots, s_n) \in T^n | \forall i \in I \; s_i = 1\}.$$

Homotopical type of strata: notations

- Ω_{ss} is a real algebraic curve without self-intersections.
- $Q = \{Q_1, \dots, Q_r\}$ is a set of connected components.
- $\Omega_s = \sqcup_{u \in U} u$ be a decomposition of Ω_s into connected components. $\Omega_{un} = \sqcup_{v \in V} v$ decomposition of Ω_{un}.
- $q(u), u \in U \cup V$ is a number of connected components of Ω_s, Ω_{un} having a common border with u.
- $\lambda = (\lambda_1, \dots, \lambda_h)$ is a partition of n.
- $(k, l, m)_\Omega, k + l + m = n$ is a strata.
- F_k is a free group with k generators.

Topology of strata: homotopy type

- $(k, l, m)_\Omega$ decomposes into union of connected components homeomorphic to $R = \prod_{i \in h_R \subseteq Q} i^{(\lambda_i)} \prod_{i \in t_R \subseteq U} i^{(\lambda_i)} \prod_{i \in w_R \subseteq V} i^{(\lambda_i)}$.
- Here $\sum_{i \in h_R} \lambda_i = l$, $\sum_{i \in t_R} \lambda_i = k$, $\sum_{i \in w_R} \lambda_i = m$.
- R varies over all possible triples of partitions.
- R is homotopically equivalent to

$$(S^1)^{|h_R| + \sum_{i \in t_R \cup w_R, 1 < q(i) \leq \lambda_i + 1} (q(i)-1)} \times \prod_{i \in t_R \cup w_R, q(i) > \lambda_i + 1} T_{\lambda_i}^{q(i)-1} \times \prod_{i \in t_R \cup w_R, \lambda_i = 1, q(i) > 2} \bigvee_{j=1}^{q(i)-1} S^1$$

- Fundamental group of R is

$$\mathbb{Z}^{|h_R| + \sum_{i \in t_R \cup w_R, \lambda_i > 1} (q(i)-1)} \times \prod_{i \in t_R \cup w_R, \lambda_i = 1, q(i) > 2} F_{q(i)-1}.$$

Standart theories

Lemma on standart theories

Let S be a stability theory. Let, moreover, following conditions holds

1. Ω_{ss} is an irreducible connected real algebraic curve.
2. Inversion $\lambda \mapsto \frac{1}{\lambda}$ is an automorphism of stratified space S.
3. Complex conjugation $\lambda \mapsto \overline{\lambda}$ is an automorphism of stratified space S.
4. 0 and ∞ cannot be both stable or both unstable.

Then S is either Hurwitz stability theory, Schur stability theory or (quasi)hyperbolicity theory(with Ω_{ss} as a real line).

Main ideas of the proof

1. Note that we can use an invariance under conjugation and conjugate of inversion.
2. Go to polar coordinates
3. Defining polynomial is either palindromic or antipalindromic.
4. Apply I.Markovsky-S.Rao(2008) results on structure of (anti)palindromic polynomials.
5. Use irreducibility and Artin-Hilbert theorem on representation as sum of squares of rational functions.

Generalisation of standart theories

Let us drop connectedness and $0-\infty$ division. What kind of regions we will have?
General equation of non-standart Ω_{ss} is an even degree polynomial:

$$\sum_{i=0}^{\frac{n}{2}}\Big(\sum_{j=0}^{\lfloor \frac{i}{2} \rfloor} a_{ij}x^{i-2j}(x^2+y^2)^j\Big)\Big(1+(x^2+y^2)^{\frac{n}{2}-i}\Big)$$

Examples of families