Gram Spectrahedra

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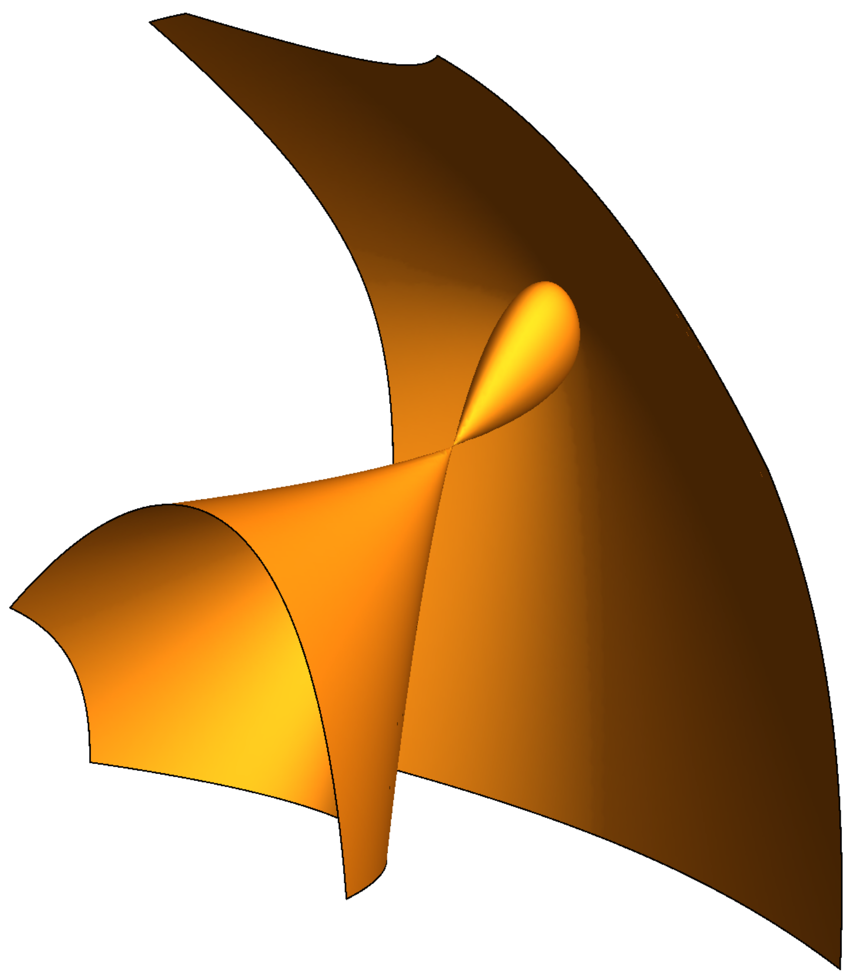

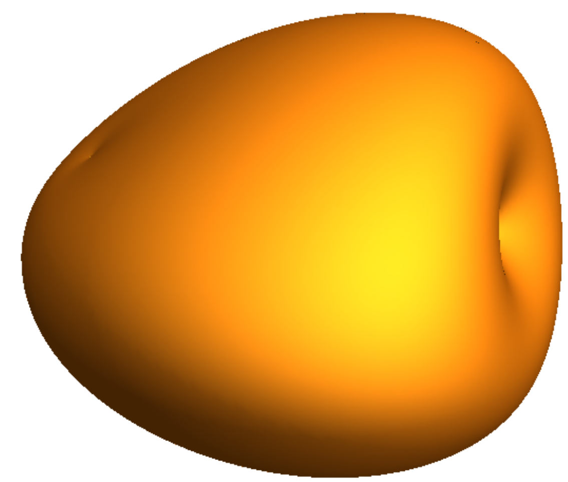

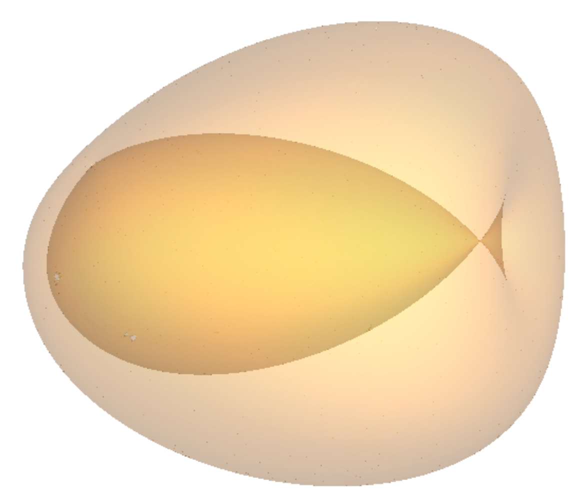

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Motivation

Algorithmic approach to sum of squares decompositions of polynomials (-> semidefinite programming and polynomial optimization)

Example. The polynomial $f = x^2 + 2x + 3$ is positive semi-definite.

(1) A quadratic polynomial has a unique Gram matrix

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}, \text{ i.e. } f = \begin{pmatrix} 1 & x \end{pmatrix} A \begin{pmatrix} 1 \\ x \end{pmatrix}.$$

This matrix is positive semidefinite because $\mathrm{tr}(A) = 4$ and $\det(A) = 2$.

(2) Alternatively,

$$f = (x+1)^2 + (\sqrt{2})^2.$$

Miracle?

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \sqrt{2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \sqrt{2} & 0 \end{pmatrix}$$

Gram Map

Consider the map

$$J\colon \begin{cases} \mathrm{Sym}_{d+1}(\mathbb{R}) & \to & \mathbb{R}[x]_{2d} \\ A & \mapsto & (1,x,\dots,x^d)A(1,x,\dots,x^d)^t \end{cases}$$

Theorem. *A polynomial $f \in \mathbb{R}[x]_{2d}$ is a sum of squares if and only if there is a positive semidefinite matrix A with $J(A) = f$.*

Proof. If $f = f_1^2 + f_2^2 + \dots + f_r^2$, then

$$A = c_1^t c_1 + c_2^t c_2 + \dots + c_r^t c_r,$$

where $f_i = c_i(1,x,\dots,x^d)^t$, will do.

Conversely, use Cholesky (type) factorization (or Principal Axis Theorem):
Let A be a positive semidefinite symmetric $(d+1)\times(d+1)$ matrix. Then there is an $r\times(d+1)$ matrix B, where r is the rank of A, with

$$A = B^t B. \qquad \blacksquare$$

In more variables, the same argument is valid, just use a vector of all monomials of degree at most d instead of $(1,x,\dots,x^d)$.

Gram Spectrahedra

Definition. The Gram spectrahedron $\mathcal{G}_f$ of a polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]_{2d}$ is the set of positive semidefinite matrices in $J^{-1}(f)$.

Properties. *(1) $\mathcal{G}_f$ is a convex set (a slice of the cone of positive semidefinite matrices with an affine linear space - a spectrahedron)*

(2) $\mathcal{G}_f$ is non-empty if and only if f is a sum of squares of polynomials.

(3) $\mathcal{G}_f$ parametrizes all representations of f as a sum of squares up to orthogonal equivalence.

(4) The rank of a Gram matrix is the minimal length of the corresponding representation of f as a sum of squares (over $\mathbb{R}$).

Consider

$$f = (1 + x + 4x^2)^2 + (x + 2x^2)^2 + (1 + 2x^2)^2 = 24x^4 + 12x^3 + 14x^2 + 2x + 2$$

This representation as a sum of squares is equivalent to a representation of length 2, namely

$$f = \frac{1}{2}(x-1)^2 + \frac{1}{2}(\sqrt{3} + \sqrt{3}x + 2\sqrt{12}x^2)^2.$$

Question: How "special" are Gram spectrahdra?

Minimal Ranks

What is the minimal rank of a positive semidefinite Gram matrix of a given polynomial?
Generic Answer: Pataki range (dimension count) – The rank r of an extreme point of a generic m-dimensional slice of the cone of positive semidefinite $k \times k$ matrices satisfies the inequalities

$$\binom{k-r+1}{2} \leq m \text{ and } \binom{r+1}{2} \leq \binom{k+1}{2} - m.$$

Gram spectrahedra:

(1) Univariate Polynomials: The ranks of extreme points of a polynomial f satisfy the Pataki inequalities if and only if f is not a square.

(2) Bivariate quartics: By a theorem of Hilbert, every nonnegative quartic polynomial in two variables is a sum of three squares, which is the lower bound of the Pataki range in this case.

Gram spectrahedra are Pataki general in all cases in which every positive polynomial is a sum of squares.

Generalization of Hilbert's Theorem on Ternary Quartics

Let $X \subset \mathbb{P}^n$ be a nondegenerate irreducible variety and assume that $X(\mathbb{R})$ is Zariski dense in X.

Theorem (Blekherman, Smith, Velasco). *Every quadratic that is nonnegative on X is a sum of squares of linear forms in $\mathbb{R}[X]$ if and only if X is of minimal degree.*

We say that X is of minimal degree if $\deg(X) = \operatorname{codim}(X) + 1$.

Example. Let $C_3 = \{(s^3 : s^2t : st^2 : t^3) \in \mathbb{P}^3 \colon (s : t) \in \mathbb{P}^1\}$ the twisted cubic in $\mathbb{P}^3 = \{(x_0 : x_1 : x_2 : x_3)\}$. Then $x_0^2 + x_3^2$ restricted to C_3 is the binary sextic $(s^3)^2 + (t^3)^2$, which is a sum of two squares.

Case of binary forms: The rational normal curve C_d of degree d is a variety of minimal degree in $\mathbb{P}^d$.

Case of ternary quartics: The quadratic Veronese surface in $\mathbb{P}^5$ is a variety of minimal degree.

By the classification of varieties of minimal degree, there is one more type of varieties of minimal degree, namely rational normal scrolls. They correspond to biforms of bidegree $(m, 2)$ studied by Choi, Lam, and Reznick.

$$p(y, z, x_1, \dots, x_n) = \sum_{i,j} a_{ij}(y, z) x_i x_j.$$

Choi, Lam, Reznick: Every nonnegative biform of bidegree $(m, 2)$ in $n + 2$ variables is a sum of $2n$ squares.

Sums of Squares on Varieties of Minimal Degree

Theorem. *Let $X \subset \mathbb{P}^n$ be a nondegenerate irreducible variety of minimal degree with dense real points. Then every quadratic nonnegative on X is a sum of $(\dim(X)+1)$ squares of linear forms in $\mathbb{R}[X]$.*

Consider the map

$$\begin{aligned} \phi\colon \mathbb{R}[X]_1 \times \mathbb{R}[X]_1 \times \cdots \times \mathbb{R}[X]_1 &\to \mathbb{R}[X]_2 \\ (\ell_1, \ell_2, \ldots, \ell_{m+1}) &\mapsto \sum_{i=1}^{m+1} \ell_i^2. \end{aligned}$$

Want to show: The image of ϕ is the cone of nonnegative quadratics on X.

The differential of ϕ is

$$\begin{aligned} \mathrm{d}\phi\colon \mathbb{R}[X]_1 \times \mathbb{R}[X]_1 \times \cdots \times \mathbb{R}[X]_1 &\to \mathbb{R}[X]_2 \\ (h_1, h_2, \ldots, h_{m+1}) &\mapsto \sum_{i=1}^{m+1} h_i \ell_i. \end{aligned}$$

The image of $\mathrm{d}\phi$ has dimension

$$(n+1)(m+1) - \text{relations} = (n+1)(m+1) - \binom{m+1}{2}$$

for generic $\ell_1, \ldots, \ell_{m+1}$, because $\ell_1, \ldots, \ell_{m+1}$ is a regular sequence and X is arithmetically Cohen-Macaulay.

Fact: $\dim_{\mathbb{R}}(\mathbb{R}[X]_2) = (n+1)(m+1) - \binom{m+1}{2}$

Sums of Squares on Varieties of Minimal Degree

Theorem. *Let $X \subset \mathbb{P}^n$ be a nondegenerate irreducible variety of minimal degree with dense real points. Then every quadratic nonnegative on X is a sum of $(\dim(X)+1)$ squares of linear forms in $\mathbb{R}[X]$.*

So the map $\phi\colon \mathbb{R}[X]_1 \times \mathbb{R}[X]_1 \times \cdots \times \mathbb{R}[X]_1 \to \mathbb{R}[X]_2$, $(\ell_1, \ell_2, \ldots, \ell_{m+1}) \mapsto \sum_{i=1}^{m+1} \ell_i^2$, is surjective onto a neighbourhood of a positive nonsingular quadratic, which is the sum of $m+1$ squares.

We can now finish the proof by the same topological argument as in Hilbert's original proof: The map ϕ is closed and every smooth quadratic in the image of ϕ is an interior point of its image. Since the set of all smooth nonnegative quadratics on X is connected, it must be equal to the image of ϕ.

It follows that every nonnegative quadratic on X is a sum of $(\dim(X)+1)$ squares. ■

Corollary. *Every nonnegative biform of bidegree $(m,2)$ in $n+2$ variables is a sum of $n+1$ squares.*

Gram spectrahedra are Pataki general in all cases in which every positive polynomial is a sum of squares.

Algebraic Boundary

Generic Case: For a generic spectrahedron, the Zariski closure of the boundary is defined by the restriction of the determinant to the affine span of the spectrahedron, which is irreducible by Bertini's Theorem.

Gram spectrahedra: They form a special family. The affine span of $\mathcal{G}_f$ is the kernel $\ker(J)$ of the Gram map shifted by a chosen Gram matrix A_f of f.

In terms of algebraic geometry, homogenizing with respect to A_f, this means that the hyperplane at infinity with respect to these coordinates is $\ker(J)$. So the projective closure of the determinant restricted to $\ker(J) + A_f$ intersected with $\ker(J)$ is the same, independent of f.

Theorem. *The restriction of the determinant to $\ker(J)$ is irreducible for $n = 1$ and $d \geq 5$ and $n \geq 2$ (and $d \geq 5$).*

Corollary. *The algebraic boundary of **every** Gram spectrahedron $\mathcal{G}_f$ is irreducible and the restriction of the determinant to the affine span of $\mathcal{G}_f$ if $\mathcal{G}_f$ has non-empty interior and f is a polynomial of degree at least 10.*

Binary Sextics and Kummer Surfaces

In this case, the Gram map is

$$J\colon \begin{cases} \mathrm{Sym}_4(\mathbb{R}) \to \mathbb{R}[x]_6 \\ A \mapsto \begin{pmatrix} 1 & x & x^2 & x^3 \end{pmatrix} A \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix} \end{cases}$$

Its kernel is

$$\begin{pmatrix} 0 & 0 & X & -Y \\ 0 & -2X & Y & Z \\ X & Y & -2Z & 0 \\ -Y & Z & 0 & 0 \end{pmatrix}.$$

The determinant restricted to the kernel is a perfect square

$$(Y^2 - XZ)^2.$$

The set of all Gram matrices of a binary form

$$f = a_6x^6 - 6a_5x^5y + 15a_4x^4y^2 - 20a_3x^3y^3 + 15a_2x^2y^4 - 6a_1xy^5 + a_0y^6$$

of degree 6 is

$$\begin{pmatrix} a_6 & -3a_5 & 3a_4 + X & -a_3 - Y \\ -3a_5 & 9a_4 - 2X & -9a_3 + Y & 3a_2 + Z \\ 3a_4 + X & -9a_3 + Y & 9a_2 - 2Z & -3a_1 \\ -a_3 - Y & 3a_2 + Z & -3a_1 & a_0 \end{pmatrix}.$$

Binary Sextics and Kummer Surfaces

Fact: The determinant of this matrix defines a **Kummer surface** in $\mathbb{P}^3$ for a generic choice of coefficients $a_0, \ldots, a_6$ of f (after homogenizing).
This Kummer surface has 16 singular points, 10 of rank 2. Of these 10 points, 4 are positive semidefinite matrices, corresponding to the four inequivalent ways of writing a binary sextic as the sum of two squares.

The dual variety of a Kummer surface is again a Kummer surface, in particular it has degree 4.
Arithmetic consequence: Let f be a positive binary sextic with rational coefficients. The entries of a generic matrix $A \in \mathcal{G}_f$ of rank 3 are algebraic numbers of degree at most 4.

In fact, this degree is only 2 because the determinant restricted to $\ker(J)$ is a perfect square, i.e. the hyperplane at infinity is tangent to the Kummer surface along a conic.

Binary Sextics and Kummer Surfaces