Extremal nonnegative forms

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Ordered Algebraic Structures and Related Topics
CIRM Luminy, 16 October 2015

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(a) We determine all sets $S \subseteq \mathbb{P}^2(\mathbf{R})$ with $|S| = 9$ for which there is $q \in P_6 \smallsetminus \Sigma_6$ with $S \subseteq Z(q)$. Call such S "admissible".

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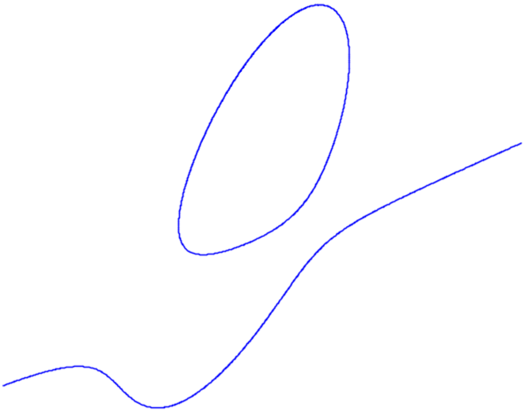
(c) For generic admissible S have $|Z(q_S)| = 10$. Thus the map $S \mapsto q_S$ is generically $10 : 1$.

Constructing admissible sets



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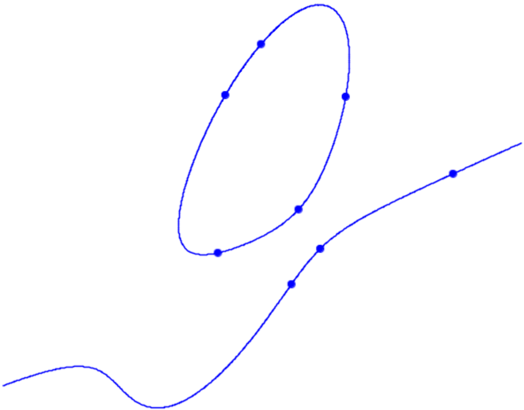

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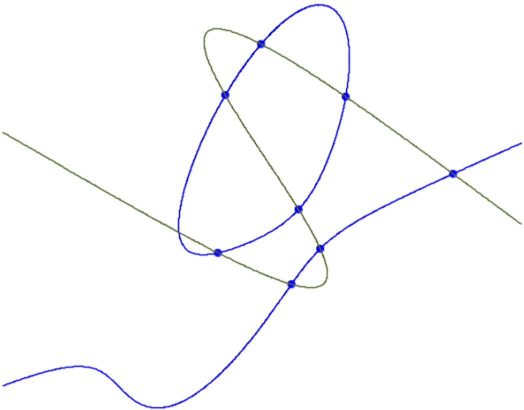

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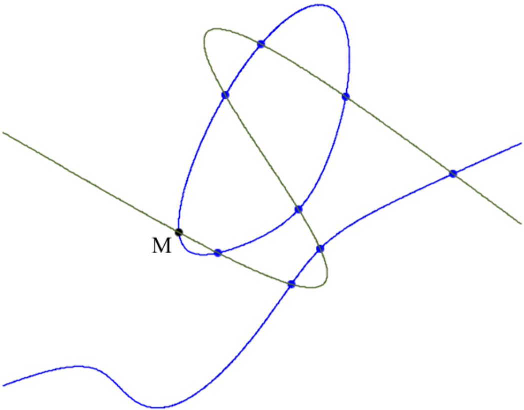

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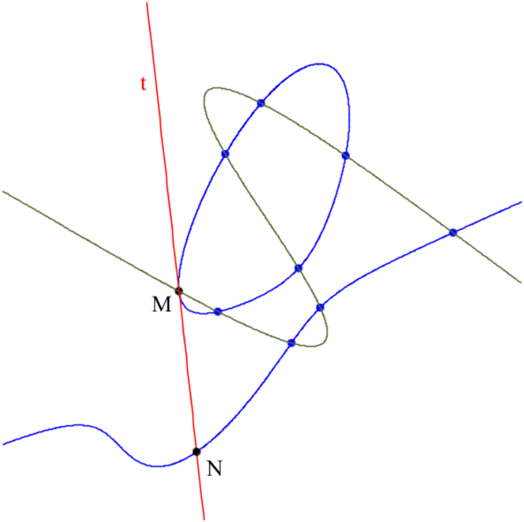

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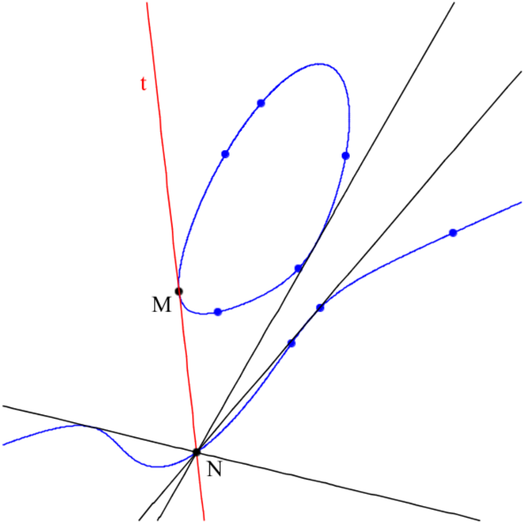

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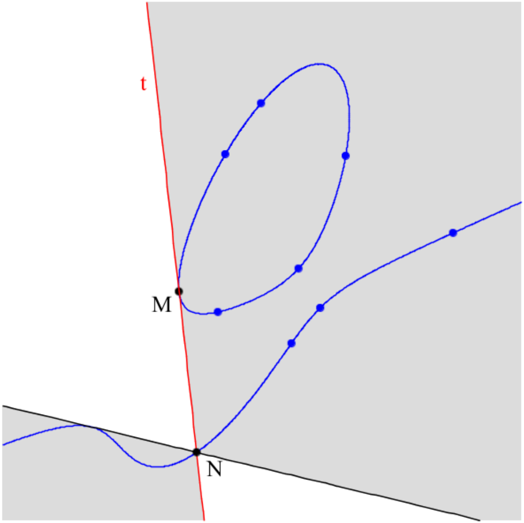

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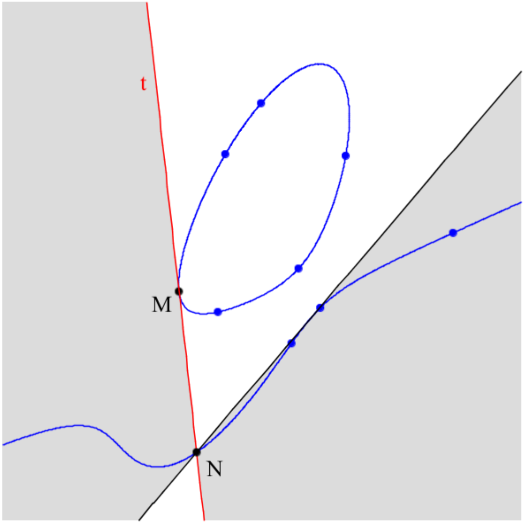

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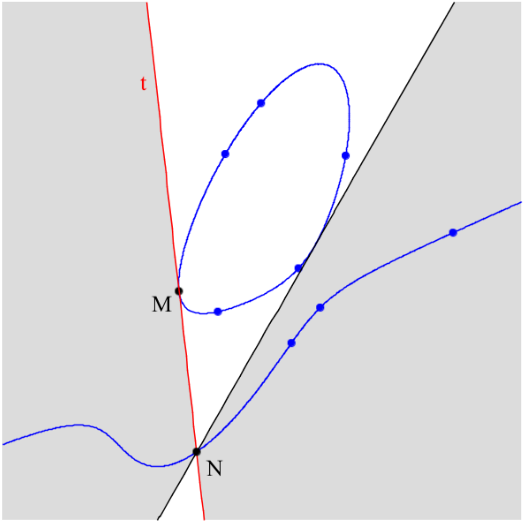

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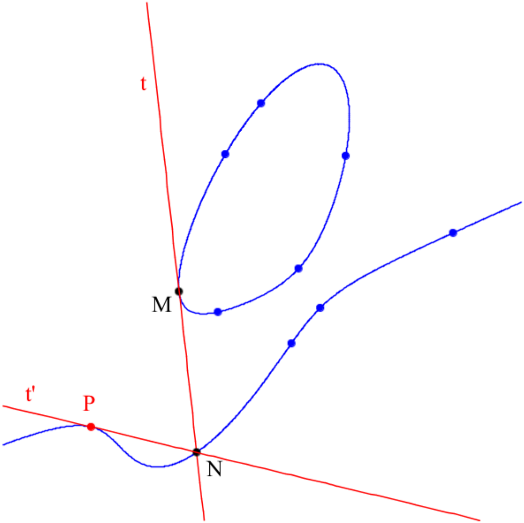

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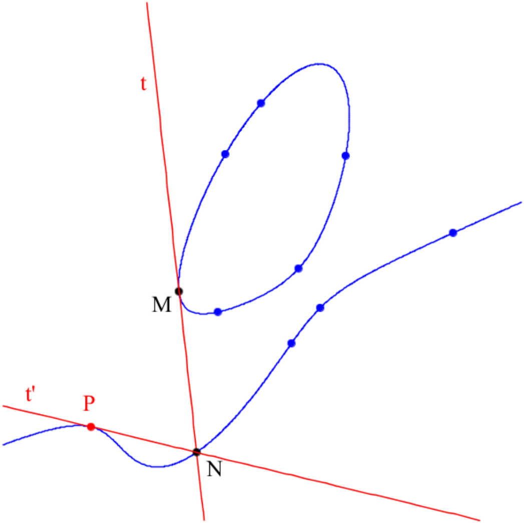

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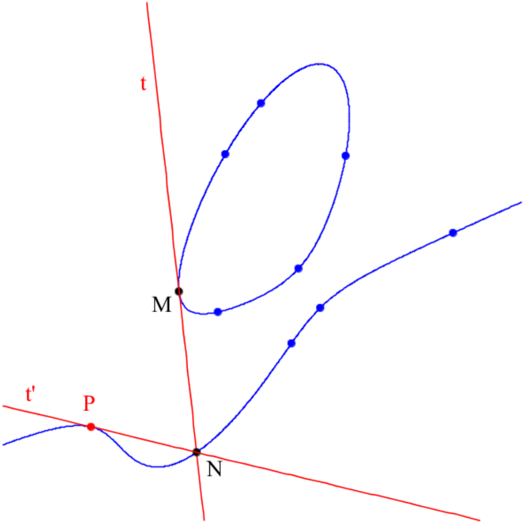

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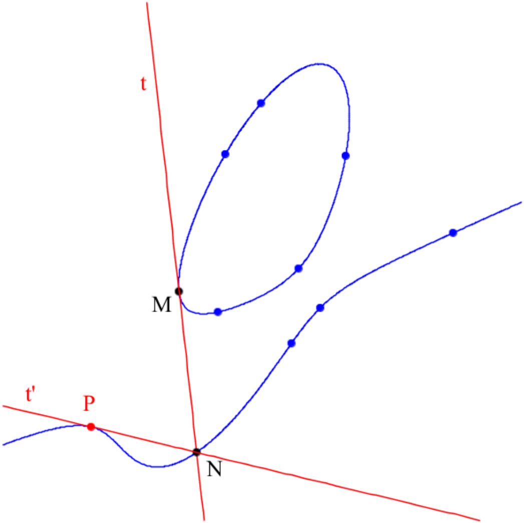

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This also works for some singular cubics X. More precisely:

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A 9-point set $S \subseteq \mathbb{P}^2(\mathbf{R})$ is admissible iff

1. $\exists$ *unique cubic X through S, and $S \subseteq X_{\mathrm{reg}}(\mathbf{R})$;*
2. *the Weil divisor $D = \sum_{P \in S} P$ on X satisfies $D \not\sim 3L$, but $2D \sim 6L$;*
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Count dimensions: 9 free parameters to choose X, plus 8 parameters to choose $S \subseteq X(\mathbf{R})$ admissible. Altogether this shows that $\mathrm{Ext}(P_6) \smallsetminus \Sigma_6$ has (projective) dimension 17 (agrees with Blekherman-Hauenstein-Ottem-Ranestad-Sturmfels 2012).

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Theorem 3
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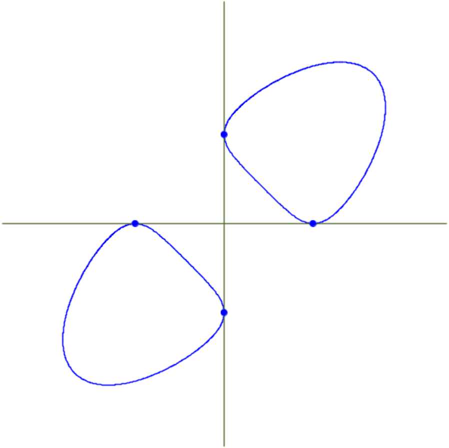
Proposition

There exists a psd octic with precisely 18 *real zeros.*

A psd octic with 18 real zeros



$f = (x^2 - 1)^2 + (y^2 - 1)^2 - 2xy - 1$
$p = 2xy$
p is tangent to f in 4 points T_1

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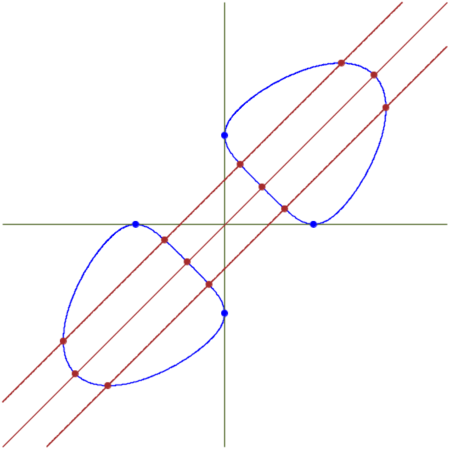

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p' intersects f in 12 points T_2

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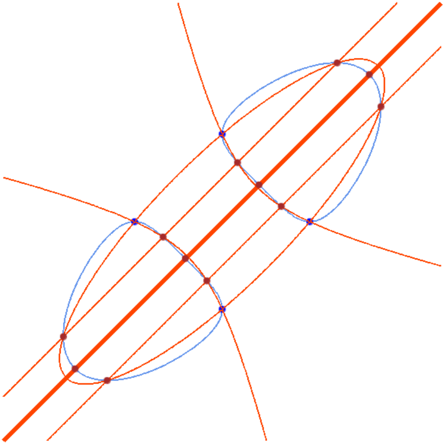

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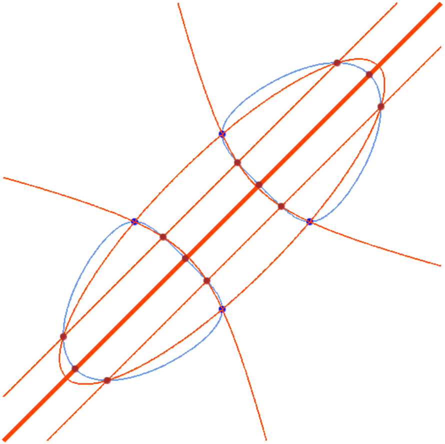

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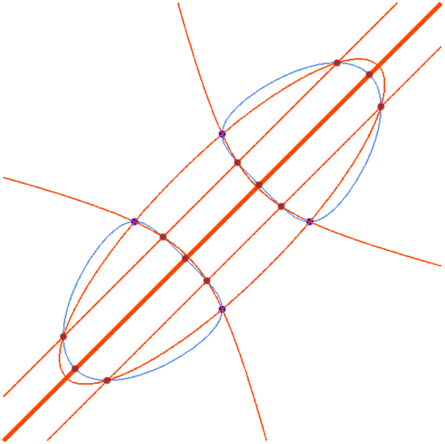

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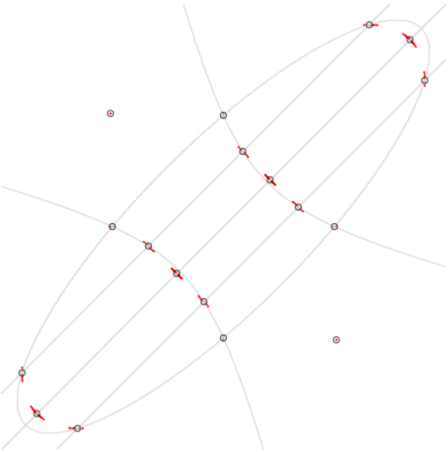

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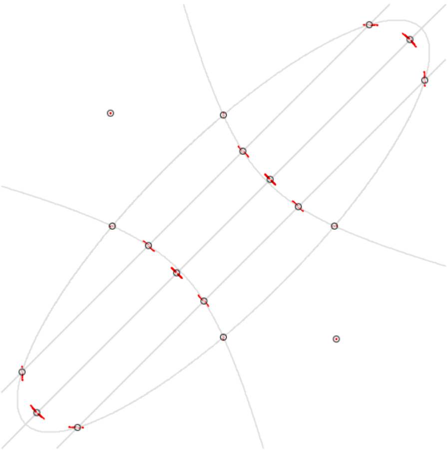

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No idea how to get 19 zeros!

Thank you DDG !
Long live DDGS !