Abstract Algebraic Theory of Quadratic Forms and Rings

F. Miraglia

Ordered Algebraic Structures and Related Topics

Luminy, October, 2015

Part 1: Approaches to the Abstract Theory of Quadratic Forms

A full-fledged approach to an abstract algebraic theory of quadratic forms, having that of fields as a model, was first developed by Murray Marshall in his work **Abstract Witt Rings** ([M1]).

This was followed by Murray's development of the theory of **Abstract Order Spaces**; a rather complete exposition of this theory can be found in [M2].

[M1] M. Marshall, **Abstract Witt Rings**, Queens Papers In Pure and Applied Math. **57**, Queen's University, Ontario, Canada.

[M2] M. Marshall, **Spaces of Orderings and Abstract Real Spectra**, Lecture Notes in Mathematics **1636**, Springer-Verlag, Berlin, 1996.

We should also mention:

• Quaternionic Structures, also presented in [M1];

• Quaternionic schemes, studied in studied by Kula, Szczepanik and Szymiczek in [KSS];

• Linked quaternionic maps, presented in [MY].

• Higher level form schemes, introduced in [MP]

[KSS] M.Kula, L.Szczepanik, K.Szymiczek, *Quadratic form schemes and quaternionic schemes*, Fund. Math., **130** (1988), 181-190.

[MY] M. Marshall, J. Yucas, *Linked Quaternionic Maps and Their Associated Witt Rings*, Pacific Jour. of Math. **95** (1981), 411-425.

[MP] M.Marshall, V.Powers, *Higher level form schemes*, Comm. Algebra, **21** (1993), 4083-4102.

The notion of **Special Group** occurred to M. Dickmann in 1991, in order to construct a 1^{st}-order theory sufficient to develop an abstract algebraic version of the theory of quadratic forms from first principles.

An exposition of the fundamentals of the ensuing theory can be found in [DM1].

Special Groups, Abstract Witt Rings, Abstract Order Spaces and the theories of quaternionic schemes mentioned above are actually **equivalent**, in the sense that models of any one system can be canonically and functorially constructed from models of the other.

[DM1] M. Dickmann, F. Miraglia, **Special Groups : Boolean-Theoretic Methods in the Theory of Quadratic Forms**, Memoirs Amer. Math. Soc. **689**, Providence, R.I., 2000.

As a final thought on the matter (time is short!), it should be registered that the interaction between abstract spaces of orderings and special groups has proven fruitful; an example – among others –, are the results in [DMM].

[DMM] M. Dickmann, M. Marshall, F. Miraglia, *Lattice ordered reduced special groups*, Annals of Pure and Applied Logic **132**, 27–49 (2005).

Part 2: Faithfully Quadratic Rings

Joint work with M. Dickmann

To appear in Memoirs of the AMS

Objective :

To lay the groundwork for a theory of quadratic forms over several significant and extensive classes of rings and preordered rings.

"Ring" stands for a commutative unitary ring where 2 is a unit;

"Quadratic forms" stands for *diagonal quadratic forms with unit coefficients*;

To this end we shall employ our **theory of special groups**, giving a unified treatment for both squares and proper preorders.

Proto-, Pre- and Special Groups

To work in the context of rings, it is convenient to split the axioms of Special Groups into certain subsets

Proto-Special Groups

A **proto-special group** **(π-SG)** is a triple

$$G = \langle\, G, \equiv_G, -1 \,\rangle,$$

such that

$*$ G is a group of exponent two, (written multiplicatively; 1 is its identity), with a distinguished element -1. Set $-x = -1 \cdot x$;

$*$ A binary relation (isometry) $\equiv_G$ on $G \times G$, such that

[SG 0] : $\equiv_G$ is an equivalence relation on $G \times G$;

[SG 1] : $\langle\, a, b \,\rangle \equiv_G \langle\, b, a \,\rangle$; [SG 2] : $\langle\, a, -a \,\rangle \equiv_G \langle\, 1, -1 \,\rangle$;

[SG 3] : $\langle\, a, b \,\rangle \equiv_G \langle\, c, d \,\rangle \quad \Rightarrow \quad ab = cd$;

[SG 5] : $\langle\, a, b \,\rangle \equiv_G \langle\, c, d \,\rangle \quad \Rightarrow \quad \langle\, xa, xb \,\rangle \equiv_G \langle\, xc, xd \,\rangle$;

G is **reduced** (π-RSG) if $1 \neq -1$ and

$$[\text{red}] : \langle\, a, a \,\rangle \equiv_G \langle\, 1, 1 \,\rangle \quad \Rightarrow \quad a = 1.$$

A π-SG, G, is a **pre-special group (p-SG)** if, in addition, it satisfies,

$$[\text{SG } 4] : \langle\, a, b \,\rangle \equiv_G \langle\, c, d \,\rangle \quad \Rightarrow \quad \langle\, a, -c \,\rangle \equiv_G \langle\, -b, d \,\rangle.$$

Let G be a π-SG. Binary isometry in G can be extended to n-forms, $n \geq 1$, still written $\equiv_G$, as follows:

$* \langle a \rangle \equiv_G \langle b \rangle \quad \Leftrightarrow \quad a = b$;

$*$ for $n = 2$, $\equiv_G$ is the primitive relation on G;

$*$ for $n \geq 3$, $\langle a_1, \ldots, a_n \rangle \equiv_G \langle b_1, \ldots, b_n \rangle$ iff there are x, y, z_3, …, $z_n \in G$ such that

(1) $\langle a_1, x \rangle \equiv_G \langle b_1, y \rangle$;

(2) $\langle a_2, \ldots, a_n \rangle \equiv_G \langle x, z_3, \ldots, z_n \rangle$;

(3) $\langle b_2, \ldots, b_n \rangle \equiv_G \langle y, z_3, \ldots, z_n \rangle$.

A p-SG is a **special group (SG)** if it verifies

[SG 6] : The isometry of forms of dimension 3 is transitive.

$*$ If G, H are π-SGs, a map $f : G \longrightarrow H$ is a **morphism** if f is a group morphism, such that $f(-1) = -1$ and

$$\langle a, b \rangle \equiv_G \langle c, d \rangle \quad \Rightarrow \quad \langle fa, fb \rangle \equiv_H \langle fc, fd \rangle.$$

Preordered rings and Proto-SGs

Preordered rings and Proto-SGs

Forthwith, all preorders are assumed to be proper; in particular, rings will be assumed to be semi-real (i.e., $-1 \notin \Sigma R^2$).

Preordered Rings

A **preordered ring** **(p-ring)** is a pair $\langle A, T \rangle$ such that A is a ring and T is a preorder of A.

$*$ The language of p-rings is $L = \langle =, +, \cdot, 0, 1, -1, T \rangle$, i.e., the 1^{st}-order language of rings , with a *unary* predicate, T, interpreted as a preorder.

$*$ A **morphism of p-rings**, $f : \langle A, T \rangle \longrightarrow \langle A', T' \rangle$, is a ring morphism, $f : A \longrightarrow A'$, such that $f(T) \subseteq T'$.

$*$ **p-Ring** is the category of p-rings and their morphisms.

p-Rings and π-SGs

To a p-ring $\langle A, T \rangle$, we associate:

$*$ A group of exponent two

$$G_T(A) \;=\; A^\times / T^\times \;=\; \{a^T : a \in A^\times\},$$

writing 1 for 1^T and -1 for $(-1)^T$;

$*$ For $a, b \in A^\times$,

$$D_T^v(a, b) \;=\; \{x \in A^\times : \exists\, s, t \in T \text{ s.t. } x = sa + tb\},$$

is the set of units **value represented** by $\langle a, b \rangle$.

$*$ Now, define

$$\langle\, a^T, b^T \,\rangle \equiv_T \langle\, c^T, d^T \,\rangle \quad \Leftrightarrow \quad \begin{cases} a^T b^T \;=\; c^T d^T \\ \text{and} \\ D_T^v(a,b) \;=\; D_T^v(c,d). \end{cases}$$

$*$ If $h : \langle\, A_1, T_1 \,\rangle \longrightarrow \langle\, A_2, T_2 \,\rangle$ is a p-ring morphism, let $h^\pi : G_{T_1}(A_1) \longrightarrow G_{T_2}(A_2)$ be given by $h^\pi(a^{T_1}) = h(a)^{T_2}$.

We have

Theorem 1

a) If $\langle A, T \rangle$ *is a p-ring, then* $G_T(A)$ *is a* π*-SG, which is reduced iff* T *is proper.*

b) If h *is a p-ring morphism,* h^π *is morphism of* π*-SGs, yielding a covariant functor from* **p-Ring** *to* $\boldsymbol{\pi}$**-SG**.

c) This functor preserves arbitrary non-empty products and all right-filtered inductive limits. ∎

$*$ The preceding construction also holds with A^2 in place of T.

In this case, write $G(A) = A^\times / {A^2}^\times$ for the associated π-SG.

$*$ If $T = \Sigma A^2$, write $G_{red}(A)$ for the associated π-SG.

Fact 2

Let A be a ring and let $T = A^2$ or a preorder of A. If A satisfies **2-transversality** *with respect to T, i.e.,*

For all $a, b \in A^\times$

$$D_T^v(a, b) =$$

$$= \{c \in A^\times : \underline{\exists\, s, t \in T^\times} \text{ s.t. } c = sa + tb\},$$

then $G_T(A)$ is a pre-special group, i.e.,

$$\langle a^T, b^T \rangle \equiv_T \langle c^T, d^T \rangle \quad \Rightarrow \quad \langle a^T, -c^T \rangle \equiv_T \langle -b^T, d^T \rangle.$$

■

These constructions hold for T-**subgroups** of $\langle A, T \rangle$, i.e., subgroups $S \subseteq A^\times$, such that $T^\times \subseteq S$ and $-1 \in S$.

If $T = A^2$, these are called q-**subgroups** of A.

Here, we treat only the case $S = A^\times$.

Our next task is to describe axioms, such that if $\langle A, T \rangle$ is a p-ring then $G_T(A)$ is a **special group**, satisfying the following requirements:

$*$ The axioms are "elementary" and closely connected to fundamental concepts of the algebraic theory of quadratic forms;

$*$ Ring-theoretic representation and isometry of forms of arbitrary dimension are faithfully coded by representation and isometry in $G_T(A)$;

$*$ In the case $T = A^2$, the mod 2 algebraic K-theory of A is naturally isomorphic to that of $G(A)$.

Diagonal $A^\times$-quadratic forms

Let $n \geq 1$ be an integer and A be a ring.

a) To an n-form over $A^\times$, $\varphi = \langle a_1, \ldots, a_n \rangle$, we associate a diagonal matrix in $GL_n(R)$, $\mathscr{M}(\varphi)$, whose non-zero entries are $a_1, \ldots, a_n$.

b) If φ, ψ are n-forms over $A^\times$, $\varphi \approx \psi$ iff

$$\exists\ M \in GL_n(R) \text{ s.t. } M\mathscr{M}(\varphi)M^t = \mathscr{M}(\psi).$$

The relation $\approx$, **matrix isometry**, is an equivalence relation.

It has the usual properties, e.g., preserves orthogonal sums and tensor products of forms.

In the case of preorders, we needed to obtain an **intrinsic** characterization of T-isometry, i.e., depending only on T, the ring operations and the ring's general linear group.

A smooth theory is forthcoming if **T-isometry** of n-forms over $A^{\times}$ is defined by:

$$\varphi \approx_T \psi$$

$$\Updownarrow$$

$\exists\ \varphi_0,\ \varphi_1, \ldots, \varphi_k$, n-dimensional forms s.t.

$(i)\ \varphi_0 = \varphi$ and $\varphi_k = \psi$;

$(ii)\ \forall\ 1 \leq i \leq k$, either $\varphi_i \approx \varphi_{i-1}$, **or**

$\varphi_i = \langle\, t_1 x_1, \ldots, t_n x_n \,\rangle$, with $t_1, \ldots, t_n \in T^{\times}$ and $\varphi_{i-1} = \langle\, x_1, \ldots, x_n \,\rangle$.

In the setting of π-SG associated to rings there are several notions of *representation* that must be distinguished. In the field case all these notions coincide.

Definition 3

Let $T = A^2$ or a proper preorder on a ring A. Let $\varphi = \langle\, b_1, \ldots, b_n \,\rangle$ and $\varphi^T = \langle\, b_1^T, \ldots, b_n^T \,\rangle$ be a $A^\times$-form and its correspondent in $G_T(A)$.

$$a)\quad D_T(\varphi) \;=\; \{a \in A^\times : \exists\; a_2, \ldots, a_n \in A^\times \text{ s. t. } \varphi^T \equiv_T \langle\, a^T, a_2^T, \ldots, a_n^T \,\rangle\}$$

are the elements **isometry-represented** *by φ^T in $G_T(A)$.* $\diamondsuit$

b) $D_T^v(\varphi) \;=\; \{a \in A^\times : \exists\, x_1, \ldots, x_n \in T$ s. t.
$$a \;=\; \textstyle\sum_{i=1}^n x_i b_i\},$$

is the set of elements **value-represented mod T by φ**.

c) $D_T^t(\varphi) \;=\; \{a \in A^\times : \underline{\exists\, z_1, \ldots, z_n \in T^\times}$ s.t.
$$a \;=\; \textstyle\sum_{i=1}^n z_i b_i\}$$

is the set of elements **transversally represented mod T by φ**.

Clearly, $D_T^t(\varphi) \subseteq D_T^v(\varphi)$.

d) Define $\mathfrak{D}_T(\varphi)$ as follows:

$*$ If $n = 2$, $\mathfrak{D}_T(\varphi) \;=\; D_T^v(b_1, b_2)$;

$*$ If $n \geq 3$,

$$\mathfrak{D}_T(\varphi) \;=\; \bigcap_{k=1}^{n} \bigcup \{D_T^v(b_k, u) : u \in D_T^v(b_1, \ldots, \overset{\vee}{b}_k, \ldots, b_n)\}.$$

"inductive value representation"

The Axioms

Let T be a preorder of a ring A or $T = A^2$.

[T-FQ 1] : (2-transversality) For all a, $b \in S$,

$$D_T^v(a, b) \;=\; D_T^t(a, b).$$

[T-FQ 2] : For all $n \geq 2$ and all n-forms φ over S,

$$D_T^v(\varphi) \;=\; \mathfrak{D}_T(\varphi).$$

[T-FQ 3] : (1-Witt-cancellation) For all integers $n \geq 1$, all $a \in A^\times$ and all n-forms φ, ψ over $A^\times$,

$$\langle\, a \,\rangle \oplus \varphi \approx_T \langle\, a \,\rangle \oplus \psi \;\Rightarrow\; \varphi \approx_T \psi.$$

We then have

Theorem 4

Let A be a ring and let T be A^2 or a preorder of A.

If $A \models$ [T-FQ 1], [T-FQ 2] *and* [T-FQ 3], *then* $G_T(A) = \langle\, G_T(A), \equiv_T, -1 \,\rangle$ *is a special group, faithfully coding T-isometry and value representation of diagonal quadratic forms over $A^\times$.* ■

I.e., if φ, ψ are n-forms over $A^\times$ and $a \in A^\times$:

• $D_T(\varphi) = D_T^v(\varphi)$, i.e., $a \in A^\times$ is value represented iff it is isometry represented in $G_T(A)$;

• $\varphi \approx_T \psi \quad \Leftrightarrow \quad \varphi^T \equiv_T \psi^T$. (in $G_T(A)$)

The preceding result has a partial converse:

Theorem 5

Let A be a ring and let T be a preorder of A, or $T = A^2$.

If $A \models$ [T-FQ 1], the following are equivalent:

(1) *$G_T(A)$ is a SG such that for all $A^\times$-forms of the same dimension, φ, ψ,*

$$(*) \quad \varphi \approx_T \psi \;\Leftrightarrow\; \varphi^T \equiv_T \psi^T;$$

$$(**) \quad D_T^v(\varphi) \;=\; D_T(\varphi^T).$$

(2) *$A \models$ [T-FQ 2] and $A \models$ [T-FQ 3].* ∎

With respect to K-theory we obtain the following general result:

Theorem 6

If A is a ring verifying 2-transversality for A^2, there is a natural graded ring isomorphism between Milnor's mod 2 K-theory of A and that of the pre-special group $G(A)$. ∎

[Gu] D. Guin, *Homologie du groupe linéaire et K-theorie de Milnor des anneaux*, J. of Algebra, **123** (1989), 27-89.

[DM3] M. Dickmann, F. Miraglia, *Algebraic K-theory of Special Groups*, Journal of Pure and Applied Algebra **204** (2006), 195-234.

The preceding results justify the following

Definition 7

Let A be a ring and T be A^2 or a preorder of A.

A is **T-faithfully quadratic** *if it satisfies axioms* [T-FQ 1], [T-FQ 2] *and* [T-FQ 3].

If $T = A^2$ write [FQ i] *for* [T-FQ i] $(i = 1, 2, 3)$, *and call A* **faithfully quadratic**. $\diamondsuit$

Let $\langle A, T \rangle$ be a p-ring. Following [KRW]

a) A **T-signature** on A is a group morphism, $\tau : A^{\times} \longrightarrow \mathbb{Z}_2 = \{\pm 1\}$, such that $\tau(-1) = -1$ and for all $a \in A^{\times}$,

$$a \in \ker \tau \quad \Rightarrow \quad D^{v}_{T}(1, a) \subseteq \ker \tau.$$

b) If $\varphi = \langle a_1, \ldots, a_n \rangle$ is a form and τ is a signature,

$sgn_{\tau}(\varphi) = \sum_{i=1}^{n} \tau(a_i)$ is the **signature** of φ at τ.

Proposition 8 (**Pfister's local global principle**)

If $\langle A, T \rangle$ is a T-faithfully quadratic ring and φ, ψ are forms of the same dimension over $A^{\times}$, then $\varphi \approx_T \psi$ iff their total signatures are the same. ■

[KRW] M. Knebusch, A. Rosenberg, R. Ware, *Signatures on Semilocal Rings*, Bulletin of the AMS **78** (1972), 62-64.

Remark: A 1^{st}-order formula is

$*$ *geometric* if it is the negation of an atomic formula **or** of the form $\forall\overline{v}(\exists\overline{y}\varphi_1(\overline{y},\overline{v};\overline{z}) \rightarrow \exists\overline{w}\varphi_2(\overline{w},\overline{v};\overline{z}))$, with φ_1, φ_2 are positive and quantifier free;

$*$ *Horn-geometric* if it is the negation of an atomic formula **or** of the form $\forall\overline{v}(\varphi_1(\overline{z}) \rightarrow \varphi_2(\overline{z}))$, where φ_1 and φ_2 are pp-formulas.

We have obtained:

• An explicit Horn-geometric axiomatization for the theory of faithfully quadratic rings in the language of unitary rings;

• An explicit geometric axiomatization for the theory of T-faithfully quadratic rings in the language of unitary p-rings.

We also have:

Theorem 9

The class of T-faithfully quadratic rings (T a proper preorder) is closed under arbitrary non-empty products and right-directed inductive limits. In particular, it is closed under arbitrary reduced products. ■

By a deep result of Kiesler, Galvin and Shelah, the theory of T-faithfully quadratic rings (T a preorder) has a Horn axiomatization.

Classes of T-faithfully Quadratic Rings

Rings with Many Units

Definition 10

Let R be a ring.

a) A polynomial $f \in R[X_1, \ldots, X_n]$ **has local unit values** *if for every maximal ideal $\mathfrak{m}$ of R, there are $u_1, \ldots, u_n$ in R such that $f(u_1, \ldots, u_n) \notin \mathfrak{m}$.*

b) R is a **ring with many units** *if for all $n \geq 1$, and all $f \in R[X_1, \ldots, X_n]$, if f has local unit values, there is $\overline{r} \in R^n$ such that $f(\overline{r}) \in R^\times$.* $\diamondsuit$

We have shown

Theorem 11

a) Rings with many units are Horn-geometric axiomatizable in the first-order language of rings.

b) The ring of formal power series in any number of variables and with coefficients in a ring with many units is again a ring with many units. $\blacksquare$

Examples of Rings with Many Units

* Fields; * semi-local rings;
* Commutative von Neumann regular rings;
* $\mathbb{C}(X) = \mathbb{C}(X, \mathbb{R})$, where X is a Boolean space.
* Theorem 11.(a) yields:

• Arbitrary non-empty reduced products and inductive limits over right-directed posets of rings with many units are rings with many units.

We now have

Theorem 12

If A is a ring with many units such that every residue field of A has at least 7 elements, then A is **completely faithfully quadratic**, *i.e., it is faithfully quadratic ($T = A^2$) and T-faithfully quadratic for any preorder T of A.* ■

Reduced f-rings

A partially ordered ring (po-ring), $\langle A, \leq \rangle$, is **lattice-ordered (lo)** if for all a, $b \in A$,

$$a \vee b \;=\; \sup\{a,\, b\} \quad \text{and} \quad a \wedge b \;=\; \inf\{a,\, b\}$$

exist in A (join and meet with respect to $\leq$).

A lo-ring is an **f-ring** if it is isomorphic to a subdirect product of linearly ordered rings.

For a lo-ring A, the following are equivalent:

(1) A is a reduced f-ring;

(2) A is a subdirect product of linearly ordered domains.

Definition 13

An f-ring, A, comes equipped with a partial order, written $T_\sharp^A$, with respect to which it is lattice ordered.

If A is clear from context, write $T_\sharp$ in place of $T_\sharp^A$. $\diamondsuit$

We then obtain, where $B(R)$ stands for the BA of idempotents of a ring R:

Theorem 14

If A is reduced f-ring and T is a preorder of A containing $T_\sharp$, then A is T-faithfully quadratic.

Moreover, $G_T(A)$ is isomorphic to $B(A)/\mathcal{I}$, where $\mathcal{I}$ is the ideal of $B(A)$, given by

$$\mathcal{I} \;=\; \{e \in B(A) : H(e) \cap Sper(A,T) = \emptyset\},$$

with $H(e) = \{\alpha \in Sper(A, T_\sharp) \colon e \notin supp(\alpha)\}$. In particular, $G_{T_\sharp}(A) = B(A)$. ∎

Some Applications

a) If X is a topological space:

(1) The ring $\mathbb{C}(X)$ is completely faithfully quadratic.

(2) If $S \subseteq \mathbb{C}(X)$ is a multiplicative set consisting of non zero-divisors, then the ring of fractions, $\mathbb{C}(X)S^{-1}$, is completely faithfully quadratic.

b) All real closed rings are completely faithfully quadratic.

By results in [KZ], item (a.2) applies to arbitrary reduced f-rings.

[KZ] M. Knebusch, D. Zhang, *Convexity, Valuations and Prüfer Extensions in Real Algebra*, Documenta Math. **10** (2005), 1-109.

Archimedean Rings with bounded inversion

Let A be a ring and let T be a preorder of A.

- T has **bounded inversion** if $1 + T \subseteq A^{\times}$. We say that $\langle A, T \rangle$ is a **BIR**. A has **weak bounded inversion (WBIR)** if $1 + \Sigma A^2 \subseteq A^{\times}$.
- T is **Archimedean** if for all $a \in A$ there is $n \in \mathbb{N}$ such that $n - a \in T$.
- A is **Pythagorean** if $A^2 = \Sigma A^2$. $\diamondsuit$

$*$ Real holomorphy rings of formally real fields are Archimedean WBIRs;

$*$ $\mathbb{C}(X)$ is a Pythagorean BIR; it is Archimedean (with its natural po) iff X is pseudo-compact.

We have

Theorem 15

If $\langle A, T \rangle$ is an Archimedean BIR and P is a preorder containing T, then A is P-faithfully quadratic.

Moreover, $G_P(A) = B(Y_P^*)$, the BA of clopens of the compact Hausdorff space of closed points of $Sper(A, P)$. ■

An important ingredient of the proof is the Becker-Schwartz version of the Kadison-Dubois Theorem.

[BS] E. Becker, N. Schwartz, *Zum Darstellungssatz von Kadison-Dubois*, Arch. Math. **40** (1983), 421-428.

Some Applications to Quadratic Form Theory over Rings and to the K-theory of Rings

Milnor's Witt Ring Conjecture

If $\langle A, T \rangle$ is a T-faithfully quadratic ring, both the Witt ring and the graded Witt ring of $\langle A, T \rangle$,

$$W_T(A) \quad \text{and} \quad \mathcal{W}_g^T(A),$$

can be constructed as usual, being naturally isomorphic to the Witt ring and graded Witt ring of the special group $G_T(A)$. In particular:

• $I_T(A) = I(G_T(A))$ is the fundamental ideal of $W_T(A)$, consisting of the classes of even dimensional forms;

• For $n \geq 1$, $I_T^n(A) = I^n(G_T(A))$ is the n^{th}-power of $I_T(A)$;

• $\mathcal{W}_g^T(A) = \langle \mathbb{F}_2, \ldots, \overline{I_T^n}(A), \ldots \rangle$ is the **graded Witt ring** of A, where for $n \geq 1$, $\overline{I_T^n}(A) = I_T^n(A)/I_T^{n+1}(A)$.

If $T = A^2$, we omit T from the notation.

Theorem 16

Let A be a Pythagorean ring, let Y^* be the subspace of closed points of $Sper(A)$. Let $B(A)$ and $B(Y^*)$ be, respectively, the BAs of idempotents of A and of clopens in Y^*.

a) If A is an f-ring, then for all $n \geq 1$,

$$k_n A \;\simeq\; \overline{I^n}(A) \;\simeq\; B(A).$$

b) If A is as Archimedean BIR, then for all $n \geq 1$,

$$k_n A \;\simeq\; \overline{I^n}(A) \;\simeq\; B(Y^*).$$

In particular, both these classes of rings satisfy Milnor's mod 2 Witt ring conjecture. ∎

The Arason-Pfister Hauptsatz

Theorem 17

Let $\langle\, A, T\,\rangle$ *be a* T*-faithfully quadratic p-ring.*

If φ *is a form over* $A^{\times}$ *such that*

$$\dim\,\varphi < 2^n \quad \textit{and} \quad \varphi \in I^n(A),$$

then, φ *is* T*-hyperbolic.*

In particular, $\bigcap_{n\geq 1} I^n(A) \;=\; \{0\}.$ ■

Marshall's Signature Conjecture

If $\langle A, T \rangle$ is a p-ring, write Y_T^* for the compact Hausdorff space of closed points in $Sper(A, T)$.

Theorem 18

a) Let $\langle A, P \rangle$ be an Archimedean BIR and let T be a preorder containing P. Let φ be a form over $A^\times$. If for some dense subset $D \subseteq Y_T^*$, we have

$$\text{For all } \beta \in D, \quad sgn_{\tau_\beta}(\varphi) \equiv 0 \text{ mod } 2^n,$$

then $\varphi \in I_T^n(A)$.

b) An analogous statement holds for an f-ring, $\langle A, T_\sharp \rangle$, and a preorder T on A so that $T_\sharp \subseteq T$. ■

A Local-global Sylvester's Inertia Law for f-rings

Theorem 19

Let A be an f-ring and let $T_\sharp$ be its natural partial order. For n-forms $\varphi = \langle\, a_1, \ldots, a_n \,\rangle$ and $\psi = \langle\, b_1, \ldots, b_n \,\rangle$ over $A^\times$, the following are equivalent:

(1) $\varphi \approx_{T_\sharp} \psi$;

(2) *There is an orthogonal decomposition of A into idempotents, $\{e_1, \ldots, e_m\}$, such that for every $1 \leq j \leq m$, the following conditions are satisfied:*

(i) *Each entry in* φ *and* ψ *is either in* $T_\sharp{}^\times e_j$ (*strictly positive in* Ae_j), *or in* $-(T_\sharp{}^\times e_j)$ (*strictly negative in* Ae_j);

(ii) *The number of entries of* φ *and* ψ *that are strictly negative in* Ae_j *is the same, i.e.*

$$\#\left(\{k \in \underline{\mathbf{n}} : a_k e_j <_{T_\sharp} 0\}\right)\} \;=\; \#\left(\{\ell \in \underline{\mathbf{n}} : b_\ell e_j <_{T_\sharp} 0\}\right). \;\blacksquare$$

$\underline{\mathbf{n}} = \{1, 2, \ldots, n\}$

That's all folks ! Many thanks for your attention !