

Moment problem for symmetric algebras of locally convex spaces

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(Joint work with M. Ghasemi, S. Kuhlmann and M. Marshall)

Ordered Algebraic Structures and Related Topics

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Outline

- 1 Introduction to the infinite dimensional moment problem
 - The classical moment problem (MP)
 - A general formulation of MP
- 2 The moment problem for symmetric algebras on a lc space
 - Formulation of the problem
 - Our results for continuous functionals
- 3 Open questions and work in progress

The classical moment problem in one dimension

Let μ be a non-negative Borel measure defined on \mathbb{R} . The n -**th moment** of μ is:

$$m_n^\mu := \int_{\mathbb{R}} x^n \mu(dx)$$

If all moments of μ exist and are finite, then $(m_n^\mu)_{n=0}^\infty$ is the **moment sequence** of μ .

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μ non-neg. Borel measure
with all moments finite



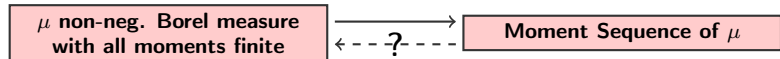
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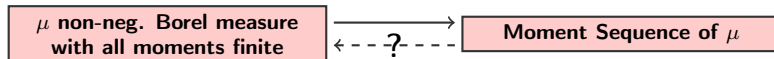


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Let $N \in \mathbb{N} \cup \{\infty\}$ and $K \subseteq \mathbb{R}$ closed.

The one-dimensional K -Moment Problem (MP)

Given a sequence $m = (m_n)_{n=0}^N$ of real numbers, does there exist a nonnegative Radon measure μ supported on a closed $K \subseteq \mathbb{R}$ s.t. for any $n = 0, 1, \dots, N$ we have

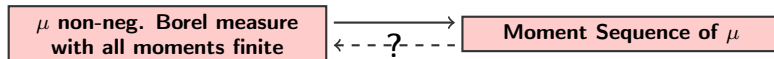
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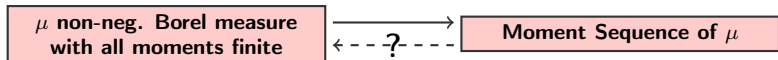
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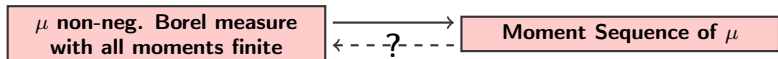
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Riesz's Functional

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Let $m = (m_n)_{n=0}^{\infty}$ be such that $m_n \in \mathbb{R}$.

$$\begin{aligned} L_m: \mathbb{R}[x] &\rightarrow \mathbb{R} \\ p(x) := \sum_{n=0}^N a_n x^n &\mapsto L_m(p) := \sum_{n=0}^N a_n m_n. \end{aligned}$$

Note:

If m is represented by a non-negative measure μ on K , then

$$L_m(p) = \sum_{n=0}^N a_n m_n = \sum_{n=0}^N a_n \int_K x^n \mu(dx) = \int_K p(x) \mu(dx).$$

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The classical K –moment problem in finite dimensions

Let $\mathbf{x} := (x_1, \dots, x_d)$ with $d \in \mathbb{N}$.

The d -dimensional K –Moment Problem (MP)

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- What if we have infinitely many variables?
- What if we take a generic \mathbb{R} –vector space V (even infinite dim.) instead of \mathbb{R}^d ?
- What if we take a \mathbb{R} –algebra A instead of the polynomial ring $\mathbb{R}[\mathbf{x}]$?

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Infinite dimensional K -Moment Problem

A general formulation of MP

Terminology and Notations:

- $A = \mathbb{R}$ -**algebra** = \mathbb{R} -vector space with a bilinear product.
- $X(A) =$ **character space** of A = the set of all ring homomorphisms $\alpha : A \rightarrow \mathbb{R}$.
- For $a \in A$ the **Gelfand transform** $\hat{a} : X(A) \rightarrow \mathbb{R}$ is $\hat{a}(\alpha) := \alpha(a)$, $\forall \alpha \in X(A)$.
- $X(A)$ is given the weakest topology s.t. all \hat{a} , $a \in A$ are continuous.

The K -moment problem for \mathbb{R} -algebras

Given a linear functional $L : A \rightarrow \mathbb{R}$, does there exist a nonnegative Radon measure μ supported on a Borel $K \subseteq X(A)$ s.t. for any $a \in A$ we have

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NB: Finite dimensional MP is a particular case

If $A = \mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \dots, x_d]$ then $X(A) = X(\mathbb{R}[\mathbf{x}])$ is identified (as tvs) with \mathbb{R}^d .
Ring homomorphisms $\mathbb{R}[\mathbf{x}] \rightarrow \mathbb{R}$ correspond to point evaluations $f \mapsto f(\alpha), \alpha \in \mathbb{R}^d$
and so $X(\mathbb{R}[\mathbf{x}])$ corresponds to \mathbb{R}^d .

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NOTE: If μ is a representing measure for L and $\text{supp}(\mu) \subseteq K$, then:

$$L(\text{Pos}(K)) \subseteq [0, +\infty) \text{ and in particular } L(M) \subseteq [0, +\infty).$$

What about the converse?

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Thm (M. Ghasemi, M. Marshall, S. Wagner 2014; M. Ghasemi, S. Kuhlmann 2013)

Let M be an archimedean $2d$ -power module of A and $L : A \rightarrow \mathbb{R}$ a linear functional.
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- M **Archimedean** if $\forall a \in A, \exists N \in \mathbb{N} : N \pm a \in M$.

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Framework

- $V = \mathbb{R}$ -vector space
- $\tau :=$ a **locally convex (lc)** topology on V
 = a topology on V generated by some family \mathcal{S} of seminorms on V
 = the weakest topology on V s.t. each $\rho \in \mathcal{S}$ is continuous.
- W.l.o.g. we assume that the family \mathcal{S} is **directed**, i.e.
 $\forall \rho_1, \rho_2 \in \mathcal{S}, \exists \rho \in \mathcal{S} \exists C > 0$ s.t. $C\rho(v) \geq \max\{\rho_1(v), \rho_2(v)\}, \forall v \in V$.
- $S(V) =$ the **symmetric algebra** of V
 = the tensor algebra $T(V)$ factored by the ideal gen. by $v \otimes w - w \otimes v$
- $S(V)_k =$ the **k -th homogeneous part** of $S(V)$
 = the image of k -th homogeneous part $V^{\otimes k}$ of $T(V)$ under the canonical map $\sum_{i=1}^n v_{i1} \otimes \cdots \otimes v_{ik} \mapsto \sum_{i=1}^n v_{i1} \cdots v_{ik}$.
- $V^* :=$ **algebraic dual** of $V = \{\ell : V \rightarrow \mathbb{R} \mid \ell \text{ is a linear functional}\}$
- $V' :=$ **topological dual** of $V = \{\ell : V \rightarrow \mathbb{R} \mid \ell \text{ is a } \tau\text{-continuous linear functional}\}$

MP for symmetric algebras on a lc space

(V, τ) with τ lc-topology. Then:

- $X(S(V)) = \text{Hom}(S(V), \mathbb{R}) \cong V^*$ via the isomorphism $\ell \mapsto \ell|_V$
- $\forall f \in S(V)$, $\hat{f} : X(S(V)) \rightarrow \mathbb{R}$ is given by $\alpha \mapsto \hat{f}(\alpha) := \alpha(f)$

The MP for symmetric algebras on a lc space

Given a linear functional $L : S(V) \rightarrow \mathbb{R}$, does there exist a nonnegative Radon measure μ on V^* s.t. for any $f \in S(V)$ we have

$$L(f) = \int_{V^*} \hat{f}(\alpha) \mu(d\alpha)?$$

Is μ unique? What is the support of μ ?

QUESTION: continuous functionals

What happens when $L : S(V) \rightarrow \mathbb{R}$ is continuous?

Which topology is natural to consider on $S(V)$? Can τ on V be extended to $S(V)$?

Continuous functionals on $S(V)$

(I. case)

(I. case): τ is generated by $S = \{\rho\}$, i.e. (V, ρ) with ρ seminorm on V .

Proposition (M. Ghasemi, M. I., S. Kuhlmann, M. Marshall, 2015)

Any seminorm ρ on V can be extended to a **submultiplicative seminorm** $\bar{\rho}$ on $S(V)$, i.e. $\bar{\rho}(fg) \leq \bar{\rho}(f)\bar{\rho}(g)$, $\forall f, g \in S(V)$.

① **tensor seminorm** $\rho^{\otimes k}$ on $V^{\otimes k}$:

$$(\rho^{\otimes k})(f) := \inf \left\{ \sum_{i=1}^n \rho(f_{i1}) \cdots \rho(f_{ik}) : f = \sum_{i=1}^n f_{i1} \otimes \cdots \otimes f_{ik}, f_{ij} \in V, n \geq 1 \right\}.$$

② Let $\pi_k : V^{\otimes k} \rightarrow S(V)_k$ be the canonical map.

For $k \geq 1$ define $\bar{\rho}_k$ to be the **quotient seminorm** $S(V)_k$ induced by $\rho^{\otimes k}$:

$$\bar{\rho}_k(f) = \inf \left\{ \sum_{i=1}^n \rho(f_{i1}) \cdots \rho(f_{ik}) : f = \sum_{i=1}^n f_{i1} \cdots f_{ik}, f_{ij} \in V, n \geq 1 \right\}.$$

Define $\bar{\rho}_0$ to be the usual absolute value on \mathbb{R} .

③ Extend ρ to a submultiplicative seminorm $\bar{\rho}$ on $S(V)$ by taking the **projective extension of ρ to $S(V)$** defined for any $f = f_0 + \cdots + f_r$, $f_k \in S(V)_k$, $k = 0, \dots, r$ by:

$$\bar{\rho}(f) := \sum_{k=0}^r \bar{\rho}_k(f_k).$$

Continuous functionals on $S(V)$

(I. case)

Proposition

(V, ρ) s.t. ρ seminorm



$(S(V), \bar{\rho})$ s.t. $\bar{\rho}$ submult. seminorm

Thm (Ghasemi, Kuhlmann, Marshall, 2014)

Let (A, σ) be a submult. seminormed \mathbb{R} -alg. and M a $2d$ -power module of A . If $L : A \rightarrow \mathbb{R}$ is a σ -continuous linear functional, then:

$(L(M) \subseteq [0, +\infty)) \Leftrightarrow (\exists ! \mu \text{ on } X(A):$

$L(a) = \int_{X(A)} \hat{a}(\alpha) \mu(d\alpha) \text{ \& supp } \mu \subseteq X_M \cap \text{sp}(\sigma))$

Theorem (M. Ghasemi, M. I., S. Kuhlmann, M. Marshall, 2015)

Let (V, ρ) be a seminormed \mathbb{R} -vector space and M be a $2d$ -power module of $S(V)$. If $L : S(V) \rightarrow \mathbb{R}$ is a $\bar{\rho}$ -continuous linear functional, then:

$(L(M) \subseteq [0, +\infty)) \Leftrightarrow (\exists ! \mu \text{ on } V^*: L(f) = \int_{V^*} \hat{f}(\alpha) \mu(d\alpha) \text{ \& supp } \mu \subseteq X_M \cap \bar{B}_1(\rho'))$

Notation:

-Gelfand spectrum of ρ : $\text{sp}(\rho) := \{\alpha \in X(A) : \alpha \text{ is } \rho\text{-continuous}\}$.

-operator norm in V^* w.r.t. ρ : $\rho'(v^*) := \inf\{C \in [0, \infty) : |v^*(f)| \leq C\rho(f) \forall f \in V\}$

-closed unitary ball in V^* w.r.t. ρ : $\bar{B}_1(\rho') := \{v^* \in V^* : \rho'(v^*) \leq 1\}$.

Continuous functionals on $S(V)$

(II. case)

(II. case): τ is a lc topology on V generated by a directed family \mathcal{S} of seminorms

Lemma

Suppose that τ is a lc topology on V generated by a directed family \mathcal{S} of seminorms.
 $(L : V \rightarrow \mathbb{R} \text{ is } \tau\text{-continuous}) \Leftrightarrow (\exists \rho \in \mathcal{S} \text{ s.t. } L \text{ is } \rho\text{-continuous}).$

Proposition (M. Ghasemi, M. I., S. Kuhlmann, M. Marshall, 2015)

(V, τ) s.t. τ lc topology $\rightarrow (S(V), \bar{\tau})$ s.t. $\bar{\tau}$ lmc topology

- $\bar{\tau}$ defined by the directed family of submultiplicative seminorms $\bar{i}\rho, \rho \in \mathcal{S}, i \in \mathbb{N}$
 - $\bar{\tau}$ is the finest lmc topology on $S(V)$ extending τ .

RECALL: A **locally multiplicatively convex** (lmc) topology on an \mathbb{R} -algebra A is a topology on A generated by some family of submultiplicative seminorms on A .

Theorem (M. Ghasemi, M. I., S. Kuhlmann, M. Marshall, 2015)

Let (V, τ) be a lc \mathbb{R} -vector space and M be a $2d$ -power module of $S(V)$.

If $L : S(V) \rightarrow \mathbb{R}$ is a $\bar{\tau}$ -continuous linear functional, then:

$(L(M) \subseteq [0, +\infty)) \Leftrightarrow (\exists ! \mu \text{ on } V^*: L(f) = \int_{V^*} \hat{f}(\alpha) \mu(d\alpha) \text{ \& \; } \text{supp } \mu \subseteq X_M \cap \bar{B}_i(\rho'))$
 for some $\rho \in \mathcal{S}$ and some integer $i \geq 1$)

Open questions and work in progress

- *Comparison* with the previous results for lc nuclear spaces.
- Can we generalize our result by *weakening the continuity* hp on $L : S(V) \rightarrow \mathbb{R}$ without any further assumption on V ?
- Would this still give a '*good*' *characterization* of the support?

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- *Comparison* with the previous results for lc nuclear spaces.
- Can we generalize our result by *weakening the continuity hp* on $L : S(V) \rightarrow \mathbb{R}$ without any further assumption on V ?
- Would this still give a '*good*' *characterization* of the support?

For more details see:



M. Ghasemi, M. Infusino, S. Kuhlmann, M. Marshall, **Moment problem for symmetric algebras of locally convex spaces**, arXiv:1507.06781.

Thank you for your attention
and

Thank you Murray

working with you was an incomparable opportunity for me.
I miss you.



Newton Institute, Cambridge-July, 2013.

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