Witt equivalence of function fields over global fields

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Hyperfields

A hyperfield is an object like a field, but where the addition is allowed to be multivalued.
A **hyperfield** is a system $(H, +, \cdot, -, 0, 1)$ where

- H is a set,
- $+$ is a function from $H \times H$ to the set $2^H$of all subsets of H,
- $\cdot$ is a binary operation on H,
- $- : H \to H$ is a function,
- $0, 1$ are elements of H

such that

I. $(H, +, -, 0)$ is a canonical hypergroup, i.e.,
  (1) $c \in a + b \Rightarrow a \in c + (-b)$,
  (2) $a \in b + 0$ iff $a = b$,
  (3) $(a + b) + c = a + (b + c)$,
  (4) $a + b = b + a$;

II. $(H, \cdot, 1)$ is a commutative monoid, i.e.,
   (1) $(ab)c = a(bc)$,
   (2) $ab = ba$,
   (3) $a1 = a$;

III. $a0 = 0$ for all $a \in H$;

IV. $a(b + c) \subseteq ab + ac$;

V. $1 \neq 0$;

VI. every non-zero element has a multiplicative inverse.

- M. Krasner, *Approximation des corps valués complets de caractéristique $p \neq 0$ par ceux de caractéristique 0*, Colloque d'algèbre supérieure, tenu à Bruxelles du 19 au 22 décembre 1956 pp. 129–206, Centre Belge de Recherches Mathématiques Établissements Ceuterick, Louvain; Librairie Gauthier-Villars, Paris (1957).
- M. Krasner, *A class of hyperrings and hyperfields*, Internat. J. Math. and Math. Sci. **6** (1983) 307–312.
- M. Marshall, *Real reduced multirings and multifields,* J. Pure and Appl. Alg. **205** (2006) 452–468.
- P. Gładki, *Orderings of higher level in multifields and multirings*, Ann. Math. Silesianae **24** (2010), 15-25.
- P. Gładki, M. Marshall, *Orderings and signatures of higher level on multirings and hyperfields*, J. K-Theory **10** (2012), 489-518.

Category of hyperfields

A **morphism** from H_1 to H_2, where H_1, H_2 are hyperfields, is a function $\alpha : H_1 \to H_2$ which satisfies

(1) $\alpha(a+b) \subseteq \alpha(a) + \alpha(b)$,

(2) $\alpha(ab) = \alpha(a)\alpha(b)$,

(3) $\alpha(-a) = -\alpha(a)$,

(4) $\alpha(0) = 0$,

(5) $\alpha(1) = 1$.

Examples of hyperfields

(A) Every field is a hyperfield (obviously);

(B) $Q_2 = \{-1, 0, 1\}$ with $\cdot$ defined in the usual way, and $+$ defined as follows:

- 0 is the neutral element of $+$,
- $1 + 1 = 1$,
- $(-1) + (-1) = (-1)$,
- $1 + (-1) = \{-1, 0, 1\}$;

this is a hyperfield.

Think of its elements as of **negative**, **zero** and **positive** reals, and of the outcome of $+$ as of adding reals with various signs.

(C) Every ordered abelian group is canonically identified with a hyperfield.
If $\Gamma := (\Gamma, \cdot, 1, \leq)$ is an ordered abelian group, the associated hyperfield is $\Gamma \cup \{0\} := (\Gamma \cup \{0\}, +, \cdot, -, 0, 1)$, where

- $a + b := \begin{cases} b \text{ if } a < b \\ a \text{ if } b < a \\ [0, a] \text{ if } a = b \end{cases}$,
- $a \cdot 0 = 0 \cdot a = 0$,
- $-a := a$.

The convention here is that $0 < a$ for all $a \in \Gamma$.

(D) Let K be a field. Recall that a **valuation** on the field K is a surjective map $v : K \to \Gamma \cup \{\infty\}$, where Γ is an ordered Abelian group, if

- $v(a) = \infty$ if and only if $a = 0$;
- $v(ab) = v(a) + v(b)$, for all $a, b \in K^*$;
- $v(a + b) \geq \min\{v(a), v(b)\}$, for all $a, b \in K^*$.

We introduce the usual notation:

- $A_v = \{a \in K : v(a) \geq 0\}$ is the valuation ring of v;
- $M_v = \{a \in K : v(a) > 0\}$ is the unique maximal ideal of A_v;
- $U_v = A_v \setminus M_v$ is the unit group of A_v;
- $K_v = A_v/M_v$ is the residue field of v.

Every valuation on a field K is just a hyperfield morphism $v : K \to \Gamma \cup \{0\}$, for some ordered abelian group $\Gamma := (\Gamma, \cdot, 1, \leq)$.

(E) If H is a hyperfield and T is a subgroup of H^*, we define the **quotient hyperfield** $H/_m T = (H/_m T, +, \cdot, -, 0, 1)$:

- $H/_m T$ is the set of equivalence classes with respect to the equivalence relation $\sim$ on H defined by

$$a \sim b \text{ if and only if } as = bt \text{ for some } s, t \in T,$$

- if $\overline{a}$ denotes the equivalence class of a, then

$$\overline{a} \in \overline{b} + \overline{c} \text{ if and only if } as \in bt + cu \text{ for some} s, t, u \in T,$$

- $\overline{a}\overline{b} = \overline{ab}$, $-\overline{a} = \overline{-a}$, $0 = \overline{0}$, $1 = \overline{1}$.

(F) Let K be a field, $K \neq \mathbb{F}_3, \mathbb{F}_5$ and $\mathrm{char}(K) \neq 2$. We will write

$$(a_1, \ldots, a_n), \text{ for } a_1, \ldots, a_n \in K^*$$

to denote the **quadratic form**

$$f(X_1, \ldots, X_n) = a_1 X_1^2 + \ldots + a_n X_n^2.$$

Moreover, denote by $D_K(a_1, \ldots, a_n)$ the value set of the form $(a_1, \ldots, a_n)$, i. e.

$$D_K(a_1, \ldots, a_n) = \{a_1 t_1^2 + \ldots + a_n t_n^2 : t_1, \ldots, t_n \in K^*\}.$$

Consider the quotient hyperfield $K/_m K^{*2}$. Then

$$a \sim b \text{ iff. } as = bt \text{ for some } s, t \in K^{*2} \text{ iff. } a = b \mod K^{*2}$$

and

$$\overline{a} \in \overline{b} + \overline{c} \text{ iff. } as \in bt + cu \text{ for some } s, t, u \in K^{*2} \text{ iff. } a \in D_K(b, c).$$

Quadratic hyperfield

Proposition

Let $H = (H, +, \cdot, -, 0, 1)$ *be a hyperfield, let the* **prime addition** *on* H *be defined as*

$$a +' b = \begin{cases} a + b & \text{if one of } a, b \text{ is zero} \\ a + b \cup \{a, b\} & \text{if } a \neq 0,\ b \neq 0,\ b \neq -a \\ H & \text{if } a \neq 0,\ b \neq 0,\ b = -a \end{cases}.$$

Then $H' := (H, +', \cdot, -, 0, 1)$ *is also a hyperfield.*

Definition

Let K be a field. The **quadratic hyperfield** of K is the quotient hyperfield $K/_m K^{*2}$ endowed with the prime addition.

Remark

(1) *Quadratic hyperfields are the same objects as quadratic form schemes with zero adjoined.*

- C. Cordes, *Quadratic forms over non-formally real fields with a finite number of quaternion algebras,* Pacific J. Math. **63** (1976), 357-365.
- M. Kula, *Fields with prescribed quadratic form schemes,* Math. Zeit. **167** (1979), 201-212.
- M. Kula, L. Szczepanik, K. Szymiczek, *Quadratic form schemes and quaternionic schemes,* Fund. Math. **130** (1988), no. 3, 181-190.

(2) Quadratic form schemes with canellation property are the same objects as quaternionic structures

- M. Marshall, *Abstract Witt rings,* Queen's Papers in Pure and Applied Math, **57**, Queen's University, Kingston, Ontario (1980).
- M. Marshall, J. Yucas *Linked quaternionic mappings and their associated Witt rings*, Pacific J. Math. **95** (1981), 411-425.
- A. Carson, M. Marshall, *Decomposition of Witt rings,* Canad. J. Math. **34** (1982), 1276-1302.

...or quaternionic schemes...

- A. Carson, M. Marshall, *Decomposition of Witt rings,* Canad. J. Math. **34** (1982), 1276-1302.

...or Abstract Witt rings...

- M. Marshall, *Abstract Witt rings,* Queen's Papers in Pure and Applied Math, **57**, Queen's University, Kingston, Ontario (1980).
- M. Marshall, J. Yucas *Linked quaternionic mappings and their associated Witt rings*, Pacific J. Math. **95** (1981), 411-425.
- A. Carson, M. Marshall, *Decomposition of Witt rings,* Canad. J. Math. **34** (1982), 1276-1302.

...or special groups

- M. Dickmann, F. Miraglia, *Special Groups : Boolean-Theoretic Methods in the Theory of Quadratic Forms,* Memoirs Amer. Math. Soc., **689,** Amer. Math. Soc., Providence, RI (2000).

(3) Real reduced hyperfields are the same objects as spaces of orderings

- M. Marshall, *Classification of finite spaces of orderings,* Canad. J. Math. **31** (1979), 320-330.
- M. Marshall, *Quotients and inverse limits of spaces of orderings,* Canad. J. Math. **31** (1979), 604-616.
- M. Marshall, *The Witt ring of a space of orderings,* Trans. Amer. Math. Soc. **258** (1980), 505-521.
- M. Marshall, *Spaces of orderings IV,* Canad. J. Math. **32** (1980), 603-627.
- M. Marshall, *Spaces of orderings: systems of quadratic forms, local structures and saturation*, Comm. in Alg. **12** (1984), 723-743.

...or real reduced special groups

- M. Dickmann, F. Miraglia, *Special Groups : Boolean-Theoretic Methods in the Theory of Quadratic Forms,* Memoirs Amer. Math. Soc., **689,** Amer. Math. Soc., Providence, RI (2000).

Witt equivalence

Denote by $W(K)$ the Witt ring of non-degenerate symmetric bilinear forms over the field K.

For K with $\operatorname{char} K \neq 2$ this is the same as Witt ring of quadratic forms.

Two fields K_1 and K_2 are **Witt equivalent** if $W(K_1) \cong W(K_2)$.

A hyperfield isomorphism $\alpha : Q(K_1) \to Q(K_2)$ can be viewed as a group isomorphism $\alpha : K_1^*/K_1^{*2} \to K_2^*/K_2^{*2}$ such that $\alpha(-\overline{1}) = -\overline{1}$ and

$$\alpha(D_{K_1}(\overline{a}, \overline{b})) = D_{K_2}(\alpha(\overline{a}), \alpha(\overline{b})) \text{ for all } \overline{a}, \overline{b} \in K_1^*/K_1^{*2}.$$

Proposition

$W(K_1) \cong W(K_2)$ as rings if and only if $Q(K_1) \cong Q(K_2)$ as hyperfields.

- D.K. Harrison, *Witt rings,* University of Kentucky Notes, Lexington, Kentucky (1970).
- J.K. Arason, A. Pfister, *Beweis des Krullschen Durchschnittsatzes für den Wittring,* Invent. Math. **12** (1971), 173-176.
- R. Baeza, R. Moresi, *On the Witt-equivalence of fields of characteristic 2,* J. Algebra **92** (1985), 446-453.

We shall denote by $K_1 \sim K_2$ two fields that are Witt equivalent.

Local fields

Local fields are complete discrete valued fields with finite residue field.

Remark

(1) *Local fields of characteristic 0 are finite extensions of* $\mathbb{Q}_p$ *(the* p*-adic competion of* $\mathbb{Q}$*).*

(2) *Local fields of characteristic* p *are fields of the form* $\mathbb{F}_{p^k}((t))$*.*

Theorem

Let (F, v) be a nondyadic local field. Then

$$F \sim \begin{cases} \mathbb{Q}_3 & \text{if } |F_v| \equiv 3 \mod 4 \\ \mathbb{Q}_5 & \text{if } |F_v| \equiv 1 \mod 4. \end{cases}$$

Theorem

Let (F, v) be a dyadic local field, and hence a finite extension of $\mathbb{Q}_2$, say $[F : \mathbb{Q}_2] = n$. Then

(1) *if n is odd, then the Witt equivalence class of F depends only on n;*

(2) *if n is even, then there are exactly 2 Witt equivalence classes:*
   - *one with $\sqrt{-1} \in F$, and*
   - *one with $\sqrt{-1} \notin F$.*

- T.-Y. Lam, *Introduction to quadratic forms over fields,* Graduate Studies in Mathematics **67** American Mathematical Society, Providence, RI (2005).

Global fields

Global fields are number fields or function fields in one variable over a finite field.

By **finite primes** of a global field we understand the discrete rank one valuations. These are the only primes in the function field case.

By **infinite primes** of a number field we understand embeddings into $\mathbb{R}$ or conjugate pairs of embeddings into $\mathbb{C}$.

Theorem

Let K_1, K_2 be global fields of characteristic $\neq 2$. Then $K_1 \sim K_2$ if and only if there exists a one-to-one correspondence between primes of K_1 and primes of K_2 (both finite and infinite) such that if $\mathfrak{p} \mapsto \mathfrak{q}$ then $\tilde{K}_{1\mathfrak{p}} \sim \tilde{K}_{2\mathfrak{q}}$, where $\tilde{K}_{1\mathfrak{p}}$ ($\tilde{K}_{2\mathfrak{q}}$) denotes the completion of K_1 (K_2) at $\mathfrak{p}$ ($\mathfrak{q}$, respectively).

- R. Perlis, K. Szymiczek, P.E. Conner, R. Litherland, *Matching Witts with global fields*, Contemp. Math. **155** (1994) 365-378.

Function fields

Theorem

Let K_1, K_2 be function fields in one variable of characteristic $\neq 2$ over algebraically closed fields k_1 and k_2, respectively. Then $K_1 \sim K_2$ if and only if $|k_1| = |k_2|$.

Theorem

Let K be an algebraic function field in one variable over a real closed field k. Then

$$K \sim \begin{cases} k(t) & \text{if } K \text{ is formally real} \\ k(t)(\sqrt{-1}) & \text{if } K \text{ is nonreal and } \sqrt{-1} \in K \\ k(t)(\sqrt{-(t^2+1)}) & \text{if } K \text{ is nonreal and } \sqrt{-1} \notin K. \end{cases}$$

- P. Koprowski, *Witt equivalence of algebraic function fields over real closed fields,* Math. Z. **242** (2002) 323-345.
- N. Grenier-Boley, D.W. Hoffmann, *Isomorphism criteria for Witt rings of real fields. With appendix by Claus Scheiderer,* Forum Math. **25** (2013) 1-18.

How about function fields over algebraic curves over other fields...?

The easiest example are function fields of rational conics.

These are of the following forms:

$$\mathbb{Q}_{a,b} := \mathrm{qf}\,\frac{\mathbb{Q}[x,y]}{(ax^2+by^2-1)} \text{ or } \mathbb{Q}_r := \mathrm{qf}\,\frac{\mathbb{Q}[x,y]}{(x^2-r)}.$$

where $a, b \in \mathbb{Q}^*$ and $r \in \mathbb{Q}^* \setminus \mathbb{Q}^{*2}$.

Using some elementary methods we were able to isolate 11 Witt non-equivalent examples of such fields.

This leads to the following natural questions:

Conjecture 1: There are 11 Witt non-equivalent function fields of rational conics.

Conjecture 2: There are infinitely many Witt non-equivalent function fields of rational conics.

Matching valuations

The main tools used in building a Witt equivalence between fields K and L rely on recognizing which valuations of K correspond to which valuations of L.

The idea here is to map some cannonical subgroups of K^*, call them T, associated to a given valuation of the field K to some subgroups of L^*, and build new valuations from there.

We say $x \in K^*$ is T-**rigid** if

$$T + Tx \subseteq T \cup Tx.$$

$$B(T) := \{x \in K^* : \text{ either } x \text{ or } -x \text{ is not } T\text{-rigid}\}.$$

Theorem

Let $H \subseteq K^*$ be a subgroup containing $B(T)$. Then there exists a subgroup $\hat{H}$ of K^* such that $H \subseteq \hat{H}$ and $(\hat{H} : H) \leq 2$ and a valuation v of K such that $1 + M_v \subseteq T$ and $U_v \subseteq \hat{H}$.

- J.K. Arason, R. Elman, W. Jacob, *Rigid elements, valuations, and realization of Witt rings*, J. Algebra **110** (1987) 449-467.

Abhyankar valuations

Let K be an algebraic function field over k.

Let v be a valuation on K.

The **Abhyankar inequality** asserts that

$$\mathrm{trdeg}(K : k) \geq \mathrm{rk}_{\mathbb{Q}}(\Gamma_v/\Gamma_{v|k}) + \mathrm{trdeg}(K_v : k_{v|k}),$$

For any abelian group Γ, $\mathrm{rk}_{\mathbb{Q}}(\Gamma) := \dim_{\mathbb{Q}}(\Gamma \otimes_{\mathbb{Z}} \mathbb{Q})$.

The valuation v is **Abhyankar** (relative to k) if

$$\mathrm{trdeg}(K : k) = \mathrm{rk}_{\mathbb{Q}}(\Gamma_v/\Gamma_{v|k}) + \mathrm{trdeg}(K_v : k_{v|k}).$$

In this case $\Gamma_v/\Gamma_{v|k} \cong \mathbb{Z} \times \cdots \times \mathbb{Z}$ with $\mathrm{rk}_{\mathbb{Q}}(\Gamma_v/\Gamma_{v|k})$ factors, and K_v is a function field over $k_{v|k}$.

- F.-V. Kuhlmann, *On places of algebraic function fields in arbitrary characteristic*, Advances in Math. **188** (2004) 399-424.

Define the **nominal transcendence degree** of K to be

$$\mathrm{ntd}(K) := \begin{cases} \mathrm{trdeg}(K : \mathbb{Q}) & \text{if } \mathrm{char}(K) = 0 \\ \mathrm{trdeg}(K : \mathbb{F}_p) - 1 & \text{if } \mathrm{char}(K) = p \neq 0 \end{cases}.$$

If K is an algebraic function field over a global field k, then $\mathrm{ntd}(K) = \mathrm{trdeg}(K : k)$.

Moreover, if v is Abhyankar (relative to k) then

$$\Gamma_v \cong \mathbb{Z} \times \cdots \times \mathbb{Z}$$

with $\mathrm{rk}_{\mathbb{Q}}(\Gamma_v)$ factors, and K_v is either a function field over a global field (if $\mathrm{ntd}(K_v) \geq 0$) or a finite field (if $\mathrm{ntd}(K_v) = -1$).

Witt equivalence of function fields over global fields

With these methods we can actually do better than just function fields of conics over $\mathbb{Q}$!

Theorem

Let K, L be function fields over global fields, $\alpha : Q(K) \to Q(L)$ a hyperfield isomorphism. Then

(1) $\mathrm{ntd}(K) = \mathrm{ntd}(L)$.

(2) *There is a canonical bijection $v \leftrightarrow w$ between Abhyankar valuations v of K with $\mathrm{ntd}(K_v) \geq 0$ and Abhyankar valuations w of L with $\mathrm{ntd}(L_w) \geq 0$ such that α maps $(1 + M_v)K^{*2}/K^{*2}$ onto $(1 + M_w)L^{*2}/L^{*2}$ and $U_v K^{*2}/K^{*2}$ onto $U_w L^{*2}/L^{*2}$.*

(3) *If $v \leftrightarrow w$ and $v' \leftrightarrow w'$ then v' is coarser than v iff w' is coarser than w.*

(4) If $v \leftrightarrow w$, v, w non-trivial, then α induces a hyperfield isomorphism $K/_m(1+M_v)K^{*2} \to L/_m(1+M_w)L^{*2}$, a hyperfield isomorphism $Q(K_v) \to Q(L_w)$, and a group isomorphism $\Gamma_v/2\Gamma_v \to \Gamma_w/2\Gamma_w$ such that the diagrams

$$\begin{tikzcd}
Q(K) \arrow[r] \arrow[d] & Q(L) \arrow[d] \\
K/_m(1+M_v)K^{*2} \arrow[r] & L/_m(1+M_w)L^{*2} \\
Q(K_v) \arrow[r] \arrow[u] & Q(K_w) \arrow[u]
\end{tikzcd}$$

and

$$\begin{tikzcd}
Q(K)^* \arrow[r] \arrow[d] & Q(L)^* \arrow[d] \\
\Gamma_v/2\Gamma_v \arrow[r] & \Gamma_w/2\Gamma_w
\end{tikzcd}$$

commute.

Corollaries to the main theorem

Let k be a number field and let r_1, respectively r_2 be the number of real embeddings of k, respectively the number of conjugate pairs of complex embeddings of k.

Thus $[k : \mathbb{Q}] = r_1 + 2r_2$.

Let

$$V_k := \{r \in k^* : (r) = \mathfrak{a}^2 \text{ for some fractional ideal } \mathfrak{a} \text{ of } k\}.$$

Clearly V_k is a subgroup of k^* and $k^{*2} \subseteq V_k$.

Corollary

Suppose $K = k(x)$, $L = \ell(x)$ where k, ℓ are number fields, and $\alpha : Q(K) \to Q(L)$ is a hyperfield isomorphism. Then

(1) *$a \in k^*/k^{*2}$ iff $\alpha(a) \in \ell^*/\ell^{*2}$.*

(2) *The map $a \mapsto \alpha(a)$ defines a hyperfield isomorphism between $Q(k)$ and $Q(\ell)$.*

(3) *α maps V_k/k^{*2} to V_ℓ/ℓ^{*2}.*

(4) *The 2-ranks of the ideal class groups of k and ℓ are equal.*

Let d be a square free integer.

The discriminant of $\mathbb{Q}(\sqrt{d})$ is d if $d \equiv 1 \mod 4$ and $4d$ otherwise.

The 2-rank of the class number of $\mathbb{Q}(\sqrt{d})$ is one less than the number of prime divisors of the discriminant of $\mathbb{Q}(\sqrt{d})$.

In particular, there are infinitely many possible values for the 2-rank of the class number for fields of the sort $\mathbb{Q}(\sqrt{d})$, $d \in \mathbb{Q}^* \backslash \mathbb{Q}^{*2}$.
Combining this with the previous Corollary, we obtain the following:

Corollary

There are infinitely many Witt inequivalent fields of the sort $\mathbb{Q}_r$, $r \in \mathbb{Q}^* \backslash \mathbb{Q}^{*2}$.