Stellensätze in closed ordered differential fields

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We want to prove analogues for differential fields of

- Dubois-Krivine-Risler Nullstellensatz,
- Stengle's Positivstellensatz,
- Schmüdgen's theorem for the real field.

Differential fields and ordered fields

Let K be a differential field.

$K\{X_1, \dots, X_n\}$ the differential ring of differential polynomials with n variables, $K\{X_1, \dots, X_n\} :=$

$$K[X_1, \dots, X_n, D(X_1), \dots, D(X_n), D^2(X_1), \dots, D^2(X_n), \dots].$$

If $p \in K\{\bar{X}\}$, $\mathrm{ord}(p) = \min\left\{k \in \mathbb{N} : p \in K[\bar{X}, D(\bar{X}), \dots, D^k(\bar{X})]\right\}$.
An ideal $I \subseteq K\{\bar{X}\}$ is differential iff

$$p \in I \Rightarrow D(p) \in I.$$

Definition

An ordered differential field is an ordered field with a derivation.
ODF denotes their first order theory in the language $\{+, -, \cdot, ^{-1}, 0, 1, <, D\}$.

Fact 1 (in ordered fields)

Let K be real closed and L an ordered field extension of K (not assumed to be real closed).
For any $g_i \in K[\bar{X}]$ and $\Box_i \in \{=, \neq, >, \geq, <, \leq\}$,
if $L \models \exists \bar{X} \bigwedge_{i=1}^{k} g_i(\bar{X}) \Box_i 0$ then $K \models \exists \bar{X} \bigwedge_{i=1}^{k} g_i(\bar{X}) \Box_i 0$.

We need a differential analogue in some theory T extending ODF.

Fact 2 (in ordered differential fields)

Let K be a model of T and L an ordered differential field extension of K (not assumed to be a model of T).
For any $g_i \in K\{\bar{X}\}$ and $\Box_i \in \{=, \neq, >, \geq, <, \leq\}$,
if $L \models \exists \bar{X} \bigwedge_{i=1}^{k} g_i(\bar{X}) \Box_i 0$ then $K \models \exists \bar{X} \bigwedge_{i=1}^{k} g_i(\bar{X}) \Box_i 0$.

T the model completion of ODF.

Singer showed that the theory ODF has a model completion, called CODF.

$f(\bar{X}) = f^*(\bar{X}, D(\bar{X}), \ldots, D^n(\bar{X}))$, where f^* is an ordinary polynomial.

Theorem 1 (Singer's axiomatisation of CODF)

Let K be an ordered differential field, $K \models CODF$ iff

1. *K is a real closed field;*
2. *for any $f, g_1, \ldots, g_m \in K\{X\}$, such that for all $i \in \{1, \ldots m\}$, $n = ord(f) \geq ord(g_i)$. If there exist $a_0, \ldots, a_n$ such that $f^*(a_0, \ldots, a_n) = 0$, $\frac{\partial}{\partial X_n} f^*(a_0, \ldots, a_n) \neq 0$ and $g_1^*(a_0, \ldots, a_n) > 0, \ldots, g_m^*(a_0, \ldots, a_n) > 0$, then there is $z \in K$ such that $f(z) = 0$ and $g_1(z) > 0, \ldots, g_m(z) > 0$.*

Definition

Let R be a ring and I an ideal of R, the real radical of I is

$$\mathcal{R}(I) := \{f \in R : f^{2m} + s \in I \text{ for some } m \in \mathbb{N} \text{ and } s \in \sum R^2\}.$$

Theorem 2 (Dubois-Krivine-Risler nullstellensatz)

Let $F \models RCF$ and I be an ideal of $F[X_1, \ldots, X_n]$, then $\mathcal{I}(\mathcal{V}(I)) = \mathcal{R}(I)$.

Theorem 3 (Nullstellensatz for CODF)

Let $K \models CODF$ and I be a differential ideal of $K\{X_1, \ldots, X_n\}$, then $\mathcal{I}(\mathcal{V}(I)) = \mathcal{R}(I)$.

Proof using:

- Model completeness of CODF
- Ritt-Raudenbush Theorem: any radical differential ideal of $K\{X_1, \ldots, X_n\}$ is finitely generated.

Proposition 4 (Grill)

Let T be a proper cone of $K\{\bar{X}\}$. The following are equivalent

1. *There is a T-convex proper differential ideal in $K\{\bar{X}\}$.*
2. *T is contained in a proper differential cone of $K\{\bar{X}\}$.*

Theorem 5 (Stengle)

Any proper differential cone is contained in a proper prime differential cone.

Theorem 6 (Positivstellensatz for CODF)

Let $K \models CODF$. Let $S \subseteq K\{X_1, \dots X_n\}$ be finite and $W_S := \{\bar{x} \in K^n : g(\bar{x}) \geq 0$ for any $g \in S\}$.
Let $f \in K\{X_1, \dots X_n\}$ and T_S be the cone of $K\{X_1, \dots X_n\}$ generated by S.
Suppose there is a T_S-convex proper differential ideal in $K\{\bar{X}\}$.

$$\forall \bar{x} \in W_S, f(\bar{x}) \geq 0 \Leftrightarrow \exists m \in \mathbb{N}, g, h \in T_S : f.g = f^{2m} + h.$$

Theorem 7

Let K be a model of CODF and S be a finite subset of $K\{\bar{X}\}$. We denote $T := T_S$ and $W := W_S$. Suppose that there is a T-convex proper differential ideal in $K\{\bar{X}\}$. Then

$$W = \varnothing \text{ iff } -1 \in T.$$

Proof.

Assume that $-1 \notin T$ and let us show that W is nonempty.
By the hypothesis, there is a T-convex proper differential ideal in $K\{\bar{X}\}$.
By Proposition 4, T is contained in a proper differential cone.
By Theorem 5, T is also contained in a proper prime differential cone P.
The support I of P is a proper prime differential P-convex ideal.
Let $L := \mathrm{Frac}(K\{\bar{X}\}/I)$, K is a differential subfield of L.

- the extension of P to L is a proper cone of L
- P is contained in an ordering P_L of L.

Taking $\bar{z} := \bar{X} + I$; one checks that for all $g \in S, g(\bar{z}) \geq 0$.
So $L \models \exists \bar{Y} \bigwedge_{g \in S} g(\bar{Y}) \geq 0$.
By model completeness of CODF (Fact 2), $K \models \exists \bar{Y} \bigwedge_{g \in S} g(\bar{Y}) \geq 0$ and so $W \neq \varnothing$. □

Topological transfers

Theorem 8 (Singer's axiomatisation of CODF)

Let K be an ordered differential field, $K \models CODF$ iff

1. *K is a real closed field;*
2. *for any $f, g_1, \ldots, g_m \in K\{X\}$, such that for all $i \in \{1, \ldots m\}$, $n = ord(f) \geq ord(g_i)$. If there exist $a_0, \ldots, a_n$ such that $f^*(a_0, \ldots, a_n) = 0$, $\frac{\partial}{\partial X_n} f^*(a_0, \ldots, a_n) \neq 0$ and $g_1^*(a_0, \ldots, a_n) > 0, \ldots, g_m^*(a_0, \ldots, a_n) > 0$, then there is $z \in K$ such that $f(z) = 0$ and $g_1(z) > 0, \ldots, g_m(z) > 0$.*

Lemma 9 (Brihaye-Michaux-Rivière)

Let $K \models CODF$. Differential tuples, i.e., tuples of the shape $(\bar{u}, D(\bar{u}), D^2(\bar{u}), \ldots, D^k(\bar{u}))$, are dense in $K^{n.k}$.

Theorem 10 (Stengle's positivstellensatz)

Let $F \models RCF$. Let $S \subseteq F[X_1, \dots, X_n]$ be finite and
$W_S := \{\bar{x} \in F^n : g(\bar{x}) \geq 0$ for any $g \in S\}$.
Let $f \in F[X_1, \dots X_n]$ and T_S be the cone of $F[X_1, \dots, X_n]$ generated by S.

$$\forall \bar{x} \in W_S, f(\bar{x}) \geq 0 \Leftrightarrow \exists m \in \mathbb{N}, g, h \in T_S : f.g = f^{2m} + h.$$

Theorem 11 (Positivstellensatz for CODF)

Let $K \models CODF$. Let $S \subseteq K\{X_1, \dots, X_n\}$ be finite and
$W_S := \{\bar{x} \in K^n : g(\bar{x}) \geq 0$ for any $g \in S\}$.
Let $f \in K\{X_1, \dots X_n\}$ and T_S be the cone of $K\{X_1, \dots X_n\}$ generated by S.
Suppose moreover that there exists an open set $O \subseteq W_S^* \subseteq \mathrm{cl}(O)$.

$$\forall \bar{x} \in W_S, f(\bar{x}) \geq 0 \Leftrightarrow \exists m \in \mathbb{N}, g, h \in T_S : f.g = f^{2m} + h.$$

Let S be a finite subset of $\mathbb{R}[\bar{X}]$ where $\bar{X} := (X_1, \ldots, X_n)$.

Theorem 12 (Schmüdgen)

If W_S is compact, then for any $f \in \mathbb{R}[\bar{X}]$,

$$(\forall \bar{x} \in W_S, f(\bar{x}) > 0) \Rightarrow f \in T_S.$$

Derivation D such that $(\mathbb{R}, D)$ is a model of CODF
Let S be a finite subset of $\mathbb{R}\{\bar{X}\}$.

Theorem 13

If W_S^* is compact and there is an open set $O \subseteq \mathbb{R}^{n \cdot (d+1)}$ such that $O \subseteq W_S^* \subseteq \mathrm{cl}(O)$, where $d := \max_{g \in S} \mathrm{ord}(g)$. Then for any $f \in \mathbb{R}\{\bar{X}\}$ of order e and any rational number q

$$(\forall \bar{x} \in W_S, f(\bar{x}) > q) \Rightarrow f \in T_E,$$

where $E := S \cup \{\pm X_i^{(j)} + r : i \in \{1, \ldots, n\}, j \in \{d+1, \ldots, e+1\}\}$ for a real number $r > 0$ (that may be chosen arbitrarily).