Mean ergodic theorem for polynomial subsequences

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Lumini, 2015

joint work with T. ter Elst

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When $\frac{1}{N}\sum_{n=1}^{N} T^{a_n}$ converges to the "proper" limit?

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Let $T \in B(H)$ *be a completely non-unitary contraction.*

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$T^n \to 0 \quad (WOT)$ is necessary

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(iii) $N^{-1}\sum_{n=1}^{N} \lambda^{a_n}$ converges for all $\lambda \in \mathbb{C}$, $|\lambda| = 1$.

Lemma

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Then

$$\lim_{N\to\infty} N^{-1} \sum_{n=1}^{N} u_n = 0.$$

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exists for each x.

The same is true for many reasonable sequences (a_n).

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for all $k \in \mathbb{N}$*.*

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(2) to some Banach spaces?

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$\sup\left\{ \frac{f^{(m+1)}(s)}{f^{(m+1)}(t)} : 1 \le t \le s \right\} < \infty$, *(for example,* $f^{(m+2)} < 0$*), and*
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Let $T \in B(H)$ be a power bounded operator on a Hilbert space H, $\sigma_p(T) \cap \mathbb{T} = \emptyset$. Let $m \in \mathbb{N}_0$, let $f : [0, \infty) \to [0, \infty)$ be a function satisfying $f' > 0$, $f'' > 0$, $\cdots, f^{(m+1)} > 0$, $\sup\left\{\frac{f^{(m+1)}(s)}{f^{(m+1)}(t)} : 1 \le t \le s\right\} < \infty$, (for example, $f^{(m+2)} < 0$), and $\sup\left\{\frac{f^{(m)}(t)}{t\, f^{(m+1)}(t)} : t \ge 1\right\} < \infty$. Let (h_n) be a bounded integer-valued sequence. Then

$$SOT - \lim_{N\to\infty} N^{-1} \sum_{n=1}^{N} T^{[f(n)]+h_n} = 0.$$

Corollary

Let $T \in B(H)$ be a power bounded operator on a Hilbert space H, $\sigma_p(T) \cap \mathbb{T} = \emptyset$.

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Corollary

Let $T \in B(H)$ *be a power bounded operator on a Hilbert space* H*,* $\sigma_p(T) \cap \mathbb{T} = \emptyset$*. Let* $f(t) = \sum_{j=0}^{k} c_j t^{\alpha_j}$*, where* $k \in \mathbb{N}_0$*,* $c_0, \dots, c_k, \alpha_0, \dots, \alpha_k \in \mathbb{R}$*,* $c_0 > 0$*,* $\alpha_0 > \max\{0, \alpha_1, \dots, \alpha_k\}$*. Then*

$$\lim_{N \to \infty} N^{-1} \sum_{n=1}^{N} T^{[f(n)]} x = 0$$

for all $x \in H$*.*

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an example is the Foguel operator

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