Linear chaos and frequent hypercyclicity

Quentin Menet

Laboratoire de Mathématiques de Lens

UNIVERSITÉ D'ARTOIS

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Plan

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T is said to be **hypercyclic** if there exists a vector $x \in X$ such that $\mathrm{Orb}(x, T) := \{T^n x : n \geq 0\}$ is dense in X.

In other words, T is hypercyclic if there exists a vector $x \in X$ such that for every non-empty open set $U \subset X$,

$$N(x, U) := \{n \geq 0 : T^n x \in U\} \neq \emptyset$$

(or equivalently, $\#N(x, U) = \infty$).

Weighted shifts

Let $w = (w_n)_{n\geq 1}$ be a bounded sequence of non-zero scalars. The weighted shift B_w is the linear and continuous operator on ℓ^p (or on c_0) given by

$$B_w(x_0, x_1, x_2, x_3, \dots) = (w_1x_1, w_2x_2, w_3x_3, w_4x_4, \dots).$$

Theorem (Salas 1995)

Let B_w be a weighted shift on c_0 or ℓ^p. Then the following assertions are equivalent :

- B_w is hypercyclic ;
- $\sup_{n\geq 1} \prod_{\nu=1}^{n} |w_\nu| = \infty$.

Periodic points

Definition

Let $T \in L(X)$. A vector $x \in X$ is a periodic point of T if there exists $d \geq 1$ such that $T^d x = x$. We denote by $\mathrm{Per}(T)$ the set of periodic points of T.

Remark

Let $T \in L(X)$. We have

$$\mathrm{Per}(T) = \mathrm{span}\{x \in X : Tx = \lambda x \text{ for some root of unity } \lambda\}.$$

Linear chaos

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1. T is hypercyclic ;
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In other words, an operator T is chaotic if in every non-empty open set,

1. there exists x such that $\mathrm{Orb}(x, T)$ is dense ;
2. there exists z such that $T^d z = z$ for some $d \geq 1$.

Example of chaotic operators

Theorem (Grosse-Erdmann 2000)

Let w be a bounded sequence of non-zero scalars. The operator B_w is chaotic on ℓ^p if and only if

$$\sum_{n=1}^{\infty} \frac{1}{\prod_{k=1}^{n} |w_k|^p} < \infty,$$

and B_w is chaotic on c_0 if and only if $\prod_{k=1}^{n} |w_k| \to \infty$.

Plan

Frequent hypercyclicity

Definition

An operator T on X is said to be **frequently hypercyclic** if there exists a vector $x \in X$ (also called frequently hypercyclic) such that for any non-empty open set $U \subset X$,

$$\underline{\text{dens}}\{n \geq 0 : T^n x \in U\} > 0,$$

where for any subset A of $\mathbb{Z}_+$ (the set of non-negative integers), we define the lower density of A as

$$\underline{\text{dens}}\, A = \liminf_{N \to \infty} \frac{\#(A \cap [0, N])}{N + 1}.$$

Frequently hypercyclic weighted shifts

Theorem (Bayart-Ruzsa 2015)

Let B_w be a weighted shift on ℓ^p. The following assertions are equivalent :

1. B_w is frequently hypercyclic ;
2. B_w is chaotic ;
3. $\sum_{k=1}^{\infty} \frac{1}{\prod_{\nu=1}^{k} |w_\nu|^p} < \infty$.

Link with ergodic theory

Let (X, T) be a linear dynamical system. If T admits an ergodic probability measure with full support, *i.e.* a probability measure m s.t.

- for every $A \in \mathcal{B}$, $m(T^{-1}A) = m(A)$,
- for every $A \in \mathcal{B}$, $T^{-1}A = A \Rightarrow m(A) = 0$ or 1,
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then, by Birkhoff ergodic theorem, we deduce that for any m-integrable function f on X, we have

$$\frac{1}{N+1}\sum_{n=0}^{N} f(T^n x) \to \int_X f dm \quad \text{for m-almost all } x \in X.$$

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By considering $f = 1_U$ where U is a non-empty open set, we get

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We conclude that T is frequently hypercyclic.

Existence of ergodic probability measures

Definition

Let X be a complex Fréchet space and $T \in L(X)$. We say that T has a **perfectly spanning set of unimodular eigenvectors** if, for every countable set $D \subset \mathbb{T}$, we have

$$\overline{\text{span}}[\ker(T - \lambda I) : \lambda \in \mathbb{T} \backslash D] = X.$$

Theorem (Grivaux 2011 - Bayart-Matheron 2014)

Let X be a complex Fréchet space and $T \in L(X)$. If T has a perfectly spanning set of unimodular eigenvectors then T admits an ergodic Gaussian measure with full support and thus T is frequently hypercyclic.

$\mathcal{U}$-frequent hypercyclicity and reiterative hypercyclicity

Definition

- An operator T on X is said to be $\mathcal{U}$-**frequently hypercyclic** if there exists a vector $x \in X$ such that for any non-empty open set $U \subset X$ $\overline{\text{dens}}\{n \geq 0 : T^n x \in U\} > 0$, where $\overline{\text{dens}}\, A = \limsup_{N\to\infty} \frac{\#(A\cap[0,N])}{N+1}$.
- An operator T on X is said to be **reiteratively hypercyclic** if there exists a vector $x \in X$ such that for any non-empty open set $U \subset X$

$$\overline{\text{Bd}}\{n \geq 0 : T^n x \in U\} > 0,$$

where $\overline{\text{Bd}}(A) = \lim_{s\to\infty} \frac{\alpha^s}{s}$ with $\alpha^s = \limsup_{k\to\infty} |A \cap [k+1, k+s]|$.

$$T \text{ fhc } \Rightarrow T\ \mathcal{U}\text{-fhc } \Rightarrow \ T \text{reit. hyp.}$$

Plan

Links between linear chaos and frequent hypercyclicity

Theorem (Bonilla - Grosse-Erdmann 2007/Bayart-Grivaux 2007)

Every chaotic weighted shift on c_0 is frequently hypercyclic but there exists a frequently hypercyclic weighted shift on c_0 which is not chaotic.

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Question

Is every chaotic operator frequently hypercyclic ?

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Every chaotic weighted shift on c_0 is frequently hypercyclic but there exists a frequently hypercyclic weighted shift on c_0 which is not chaotic.

Question

Is every chaotic operator frequently hypercyclic ?

Theorem (M. 2015)

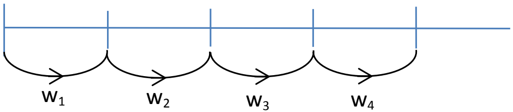
There exists a chaotic operator on ℓ^p which is not $\mathcal{U}$-frequently hypercyclic and thus not frequently hypercyclic.

Read-type operators

Read-type operators are defined on ℓ^1 by

$$Te_k = w_{k+1}e_{k+1} + f_k,$$

where $f_k \in \mathrm{span}\{e_0, \ldots, e_k\}$.



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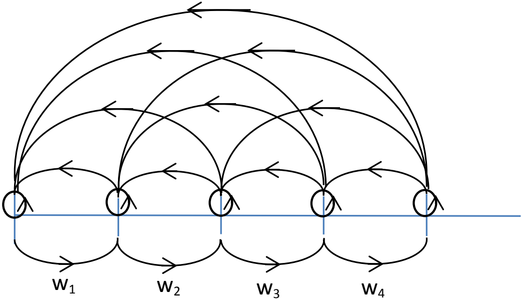


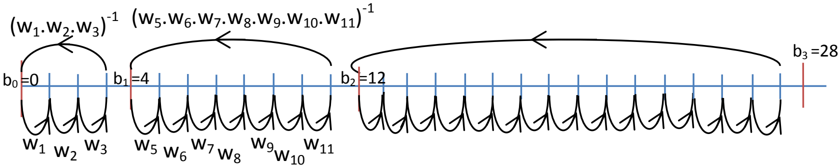
Theorem (Read 1986)

There exists a Read-type operator on ℓ^1 possessing no non-trivial invariant subspace.

Theorem (Bayart-Matheron 2007)

There exists a Read-type operator T on ℓ^1 such that $T \oplus T$ is not hypercyclic and e_0 has a dense orbit under the action of T.

Goal 1 : Find Read-type operators T such that e_k is a periodic point for T for every $k \geq 0$.



$$Te_k = \begin{cases} w_{k+1} e_{k+1} & \text{if } k \in [b_n, b_{n+1} - 1[\\ \left(\prod_{j=b_n+1}^{b_{n+1}-1} w_j\right)^{-1} e_{b_n} & \text{if } k = b_{n+1} - 1 \end{cases}$$

Remark : T is continuous if w is bounded and $w_k \geq 1$ for every k.

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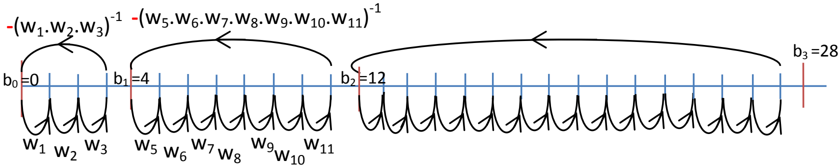

$$Te_k = \begin{cases} w_{k+1}e_{k+1} & \text{if } k \in [b_n, b_{n+1}-1[\\ -\left(\prod_{j=b_n+1}^{b_{n+1}-1} w_j\right)^{-1} e_{b_n} & \text{if } k = b_{n+1}-1 \end{cases}$$

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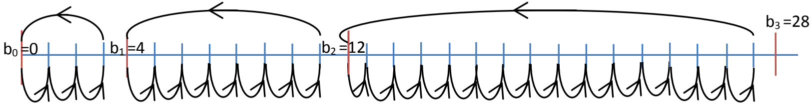
Goal 2 : Find **hypercyclic** Read-type operators T such that e_k is a periodic point for T for every $k \geq 0$.



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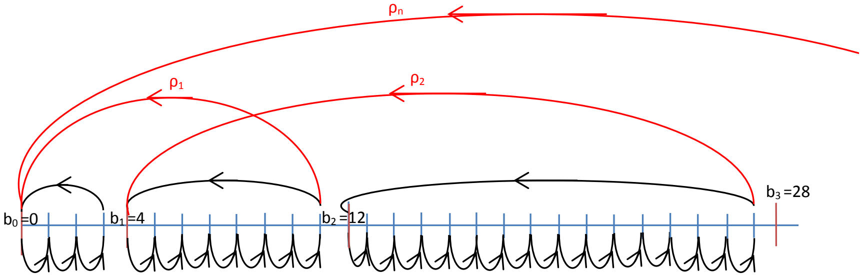


$$Te_k = \begin{cases} w_{k+1} e_{k+1} & \text{if } k \in [b_n, b_{n+1} - 1[\\ \rho_n e_{b_{\varphi(n)}} - \left(\prod_{j=b_n+1}^{b_{n+1}-1} w_j \right)^{-1} e_{b_n} & \text{if } k = b_{n+1} - 1 \quad \text{with } n \geq 1 \\ - \left(\prod_{j=1}^{b_1 - 1} w_j \right)^{-1} e_0 & \text{if } k = b_1 - 1 \end{cases}$$

Goal 2 : Find **hypercyclic** Read-type operators T such that e_k is a periodic point for T for every $k \geq 0$.

Proposition 1 (M. 2015)

Let w and ρ be two bounded sequences with $w_k \geq 1$ for every k and let $T : \ell^1 \to \ell^1$ be the operator defined by

$$Te_k = \begin{cases} w_{k+1} e_{k+1} & \text{if } k \in [b_n, b_{n+1}-1[\\ \rho_n e_{b_{\varphi(n)}} - \left(\prod_{j=b_n+1}^{b_{n+1}-1} w_j\right)^{-1} e_{b_n} & \text{if } k = b_{n+1}-1 \quad \text{with } n \geq 1 \\ -\left(\prod_{j=1}^{b_1-1} w_j\right)^{-1} e_0 & \text{if } k = b_1 - 1. \end{cases}$$

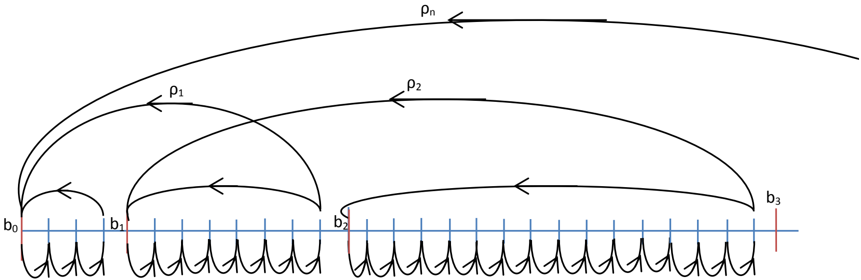
If $\varphi(n) < n$ for every $n \geq 1$ and if $(b_{n+1} - b_n)$ is a multiple of $2(b_n - b_{n-1})$ for every $n \geq 1$ then for every $n \geq 0$, every $k \in [b_n, b_{n+1}[$, we have

$$T^{2(b_{n+1}-b_n)} e_k = e_k.$$

Therefore, the operator T possesses a dense set of periodic points.

Proof : Let $n \geq 1$. We assume that $T^{2(b_{m+1}-b_m)}e_{b_m} = e_{b_m}$ is satisfied for every $m < n$. By definition of T, we get

$$T^{b_{n+1}-b_n}e_{b_n} = \Big(\prod_{j=b_n+1}^{b_{n+1}-1} w_j\Big)\rho_n e_{b_{\varphi(n)}} - e_{b_n}.$$



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Since $\varphi(n) < n$ and $b_{n+1} - b_n$ is a multiple of $2(b_{\varphi(n)+1} - b_{\varphi(n)})$, we then deduce from our induction hypothesis that

$$\begin{aligned} T^{2(b_{n+1}-b_n)}e_{b_n} &= T^{b_{n+1}-b_n}\left(\Big(\prod_{j=b_n+1}^{b_{n+1}-1} w_j\Big)\rho_n e_{b_{\varphi(n)}} - e_{b_n}\right) \\ &= \Big(\prod_{j=b_n+1}^{b_{n+1}-1} w_j\Big)\rho_n e_{b_{\varphi(n)}} - T^{b_{n+1}-b_n}e_{b_n} \\ &= \Big(\prod_{j=b_n+1}^{b_{n+1}-1} w_j\Big)\rho_n e_{b_{\varphi(n)}} - \left(\Big(\prod_{j=b_n+1}^{b_{n+1}-1} w_j\Big)\rho_n e_{b_{\varphi(n)}} - e_{b_n}\right) = e_{b_n}. \end{aligned}$$

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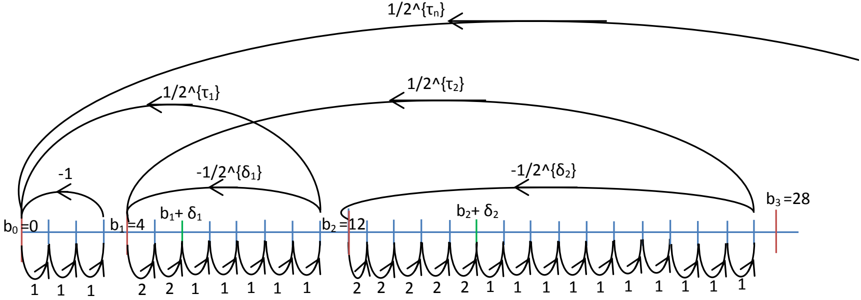
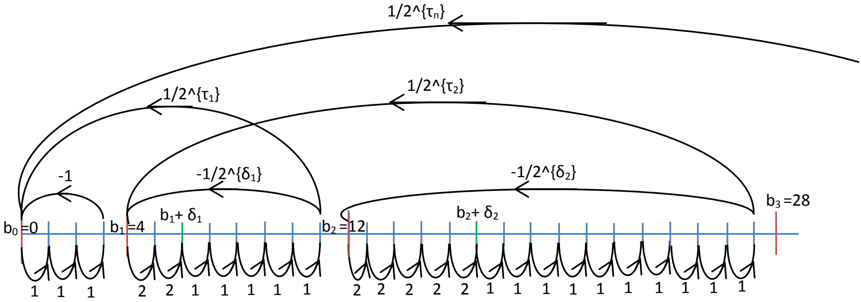


$$Te_k = \begin{cases} 2e_{k+1} & \text{if } k \in [b_n, b_n + \delta_n[\\ e_{k+1} & \text{if } k \in [b_n + \delta_n, b_{n+1} - 1[\\ \frac{1}{2^{\tau_n}} e_{b_{\varphi(n)}} - \frac{1}{2^{\delta_n}} e_{b_n} & \text{if } k = b_{n+1} - 1 \text{ with } n \geq 1 \\ -e_0 & \text{if } k = b_1 - 1 \end{cases}$$



Proposition 2 (M. 2015)

Under the assumptions of Proposition 1, if for every $k \geq 0$, $\#\varphi^{-1}(k) = \infty$ and $\delta_n - \tau_n \to \infty$ then T is chaotic on ℓ^1.

Idea : Given $k \geq 0$, there exists $n \geq 0$ as large as desired such that $\varphi(n) = k$ and thus

$$T^{b_{n+1}-b_n}\left(\frac{2^{\tau_n}}{2^{\delta_n}} e_{b_n}\right) = e_{b_k} - \frac{2^{\tau_n}}{2^{\delta_n}} e_{b_n}.$$

Goal 3 : Find a chaotic operator which is not $\mathcal{U}$-frequently hypercyclic.

Idea : It is sufficient to show that for every hypercyclic vector x for T, there exists $\varepsilon > 0$ such that

$$\overline{dens}(\{n \geq 0 : T^n x \in B(0, \varepsilon)\}) = 0.$$

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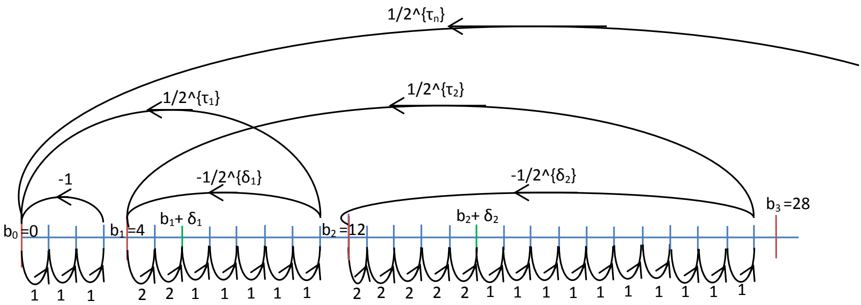
Since $\overline{\text{dens}}(A) + \underline{\text{dens}}(A^c) = 1$, it thus suffices to prove that for every hypercyclic vector x for T, there exists $\varepsilon > 0$ such that

$$\underline{dens}(\{n \geq 0 : \|T^n x\| \geq \varepsilon\}) = 1.$$



Proposition 3 (M. 2015)

Under the assumptions of Proposition 2, if $\frac{\delta_n}{b_{n+1}-b_n} \searrow 0$ and $\tau_k \geq \delta_{k-1} + 2(k+1)$ for every $k \geq 1$, then T is chaotic ℓ^1 and T is not $\mathcal{U}$-frequently hypercyclic.

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Proof : Thanks to previous propositions, we know that T is chaotic on ℓ_1 and not frequently hypercyclic if

1. $(b_{n+1} - b_n)$ is a multiple of $2(b_n - b_{n-1})$ for every $n \geq 1$;
2. $\delta_n - \tau_n \to \infty$;
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4. $\tau_k \geq \delta_{k-1} + 2(k+1)$ for every $k \geq 1$.

Each of these properties is satisfied if we consider

$$\tau_n = 4^{n+1}, \quad \delta_n = 2\tau_n \quad \text{and} \quad b_n - b_{n-1} = 4^{2n+1}.$$

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The operator T, given by the parameters

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is a chaotic operator with only countably many unimodular eigenvalues. More precisely, every unimodular eigenvalue λ of T satisfies $\lambda^{b_{n+1}-b_n} = 1$ for some $n \geq 0$.

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For every $n \geq 0$, the set $U_n := \bigcap_{l=0}^{n} T^{-ld} U$ is a non-empty open set.
In particular, for every $n \geq 0$, the set $N(x, U_n)$ is non-empty, i.e. there exists $k_n \geq 0$ such that $T^{k_n+ld} x \in U$ for every $l \leq n$.

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This implies that

$$N(x, U) \supset \bigcup_{n \geq 0} \{k_n + ld : 0 \leq l \leq n\}.$$

We conclude that T is reiteratively hypercyclic since

$$\overline{\mathrm{Bd}}\Big(\bigcup_{n \geq 0} \{k_n + ld : 0 \leq l \leq n\} \Big) \geq \frac{1}{d} > 0.$$

Thank you for your attention.