Strange Products of Orthogonal Projections

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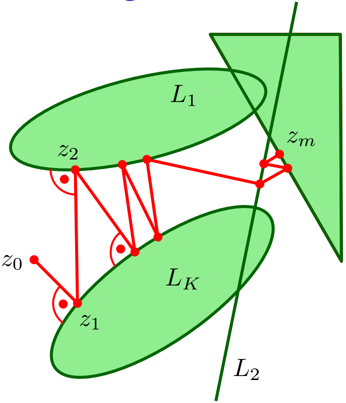
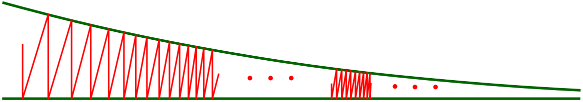
To converge, or not to converge, that is the question:



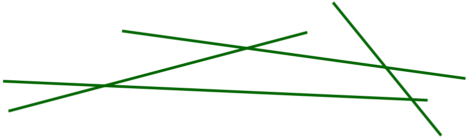
K fixed, *e.g.* $K = 5$
$L_1, L_2, \ldots, L_K \subset \mathbb{R}^d$ or ℓ_2
closed convex sets

$k_1, k_2, \cdots \in \{1, 2, \ldots, K\}$ be arbitrary
$z_n = P_{k_n} z_{n-1}$ sequence of projections

DO THE ITERATES CONVERGE?
$\equiv$ stay bounded, converge weakly, or converge in norm?



Affine subspaces



$L_1, L_2, \ldots, L_K$ closed *affine* subspaces of a Hilbert space H
$z_n = P_{k_n} z_{n-1}$ iterates of orthonormal projections of a point z

In $\mathbb{R}^d$, the sequence $\{z_n\}$ is always bounded.
[Aharoni, Duchet, Wajnryb '84], [Meshulam '96]

In ℓ_2, there exist two closed affine subspaces L_1, L_2 and a sequence $\{z_n\}$ of iterates of nearest point projections which is *not* bounded.
[Bauschke, Borwein '94]
Proof: Take two closed subspaces the sum of which is not closed, and translate one of them.

Linear subspaces

$L_1, L_2, \dots, L_K$ closed subspaces of a Hilbert space H
$z_n = P_{k_n} z_{n-1}$ iterates of orthonormal projections of a point z

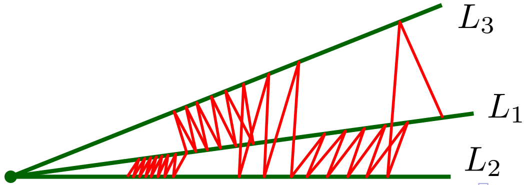
If $H = \mathbb{R}^d$, then $\{z_n\}$ converges. [Práger '60]

If $H = \ell_2$, then $\{z_n\}$ converges *weakly*. [Amemiya, Ando '65]
$(\sum_1^N z_n)/N$ converges in norm for almost all $\{k_n\} \in \{1, \dots, K\}^{\mathbb{N}}$.

If $L_1, L_2 \subset \ell_2$, then $\{z_n\}$ converges in norm. [von Neumann '49]

ASSUME $L_1, \dots, L_K \subset \ell_2$. DOES $\{z_n\}$ CONVERGE IN NORM???

Yes, if the iterates are (quasi)periodic *e.g.* $P_1 P_2 P_3 P_1 P_2 P_3 \dots$
[Halperin '62], [Sakai '95]



Projections on 3 subspaces do not have to converge

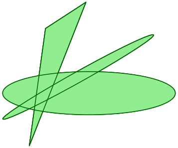
Let H be an infinite dimensional Hilbert space.

There exist 5 closed subspaces L_1, L_2, L_3, L_4, L_5 of H
and a sequence $z_n = P_{k_n} z_{n-1}$ of iterates of orthonormal
projections of a point z on L_1, L_2, L_3, L_4, L_5
which converges weakly but does not converge in norm.
[Adam Paszkiewicz, 2012]

There exist 3 closed subspaces L_1, L_2, L_3 of H
and a sequence $z_n = P_{k_n} z_{n-1}$ of iterates of orthonormal
projections of a point z on L_1, L_2, L_3
which converges weakly but does not converge in norm.
[Eva Kopecká & Vladimír Müller, 2013]

If L_1, L_2 are closed subspaces of H, then any sequence $\{z_n\}$ of
projections on L_1, L_2 converges in norm.
[John von Neumann 1949]

Convex sets



closed and convex $C_1, C_2, \ldots, C_K \subset H$
$\bigcap C_i \neq \emptyset$

$z_n = P_{k_n} z_{n-1}$ iterates of the nearest point projections of a point z

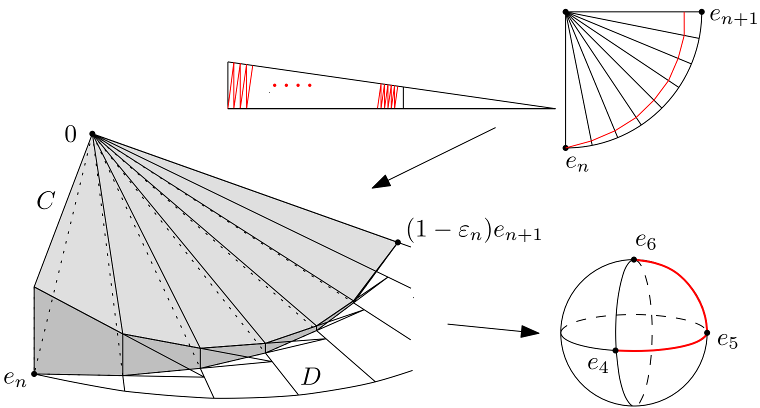
If $H = \mathbb{R}^d$ then $\{z_n\}$ converges. [Dye, Kuczumow, Lin, Reich '96]

If $K = 3$ and $H = \ell_2$ then $\{z_n\}$ converges *weakly*.
[Bruck '82], [Dye, Reich '92]
For $K = 4$ this is not known!

If $H = \ell_2$, then every sequence $\{z_n\}$ of *periodic* iterates converges *weakly*. [Bregman '65]

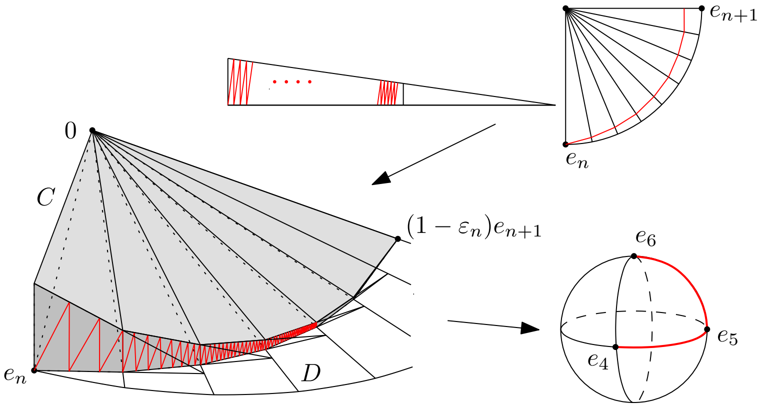
There exist $C, D \subset \ell_2$ closed and convex with $0 \in C \cap D$, and a sequence $\{z_n\}$ of iterates of nearest point projections on these sets which converges weakly but *not* in norm. [Hundal '04]

No norm-convergence in ℓ_2 already for 2 convex sets



In ℓ_2 there exist a closed convex set C, a hyperplane D, with $0 \in C \cap D$, and a point z so that the iterates $(P_C P_D)^n z$ do *not* converge in norm.
The iterates approximately contain an ON sequence $\{e_n\}$.

No norm-convergence in ℓ_2 already for 2 convex sets



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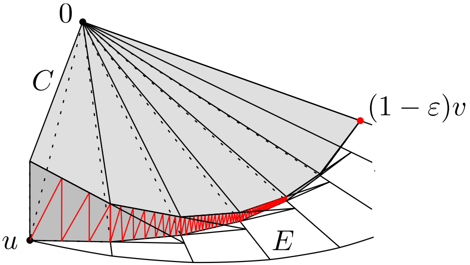
From a convex to a linear counterexample

Let u and v be orthonormal, E be the span of u and v, $\varepsilon > 0$.

[Hundal] There exists a convex cone $C \subset \mathbb{R}^3$ and a product φ of nearest point projections onto C and E so that

$$|\varphi(C, E)(u) - v| < \varepsilon.$$



[Paszkiewicz] There exist subspaces X and Y of $\mathbb{R}^d$ and a product φ of projections onto X, Y and E so that

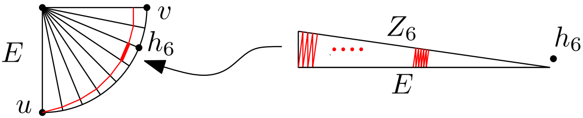
$$|\varphi(X, Y, E)(u) - v| < \varepsilon.$$

From u to v via 3 linear subspaces of $\mathbb{R}^d$

Let u and v be orthonormal, E be the span of u and v, $\varepsilon > 0$.

There exist subspaces $Z_1 \subset Z_2 \subset \cdots \subset Z_k$, $\dim Z_j = j + 1$ and a product φ of projections on these spaces and on E so that

$$|\varphi(Z_1, \ldots, Z_k, E)(u) - v| < \varepsilon.$$



Suppose $Z_1 \subset Z_2 \subset \cdots \subset Z_k = X$ are subspaces of $\mathbb{R}^d$. There is a subspace $Y \approx X$ and $m_1 > m_2 > \cdots > m_k$ so that for all $j = 1, \ldots, k$

$$\|(P_X P_Y P_X)^{m_j} - P_{Z_j}\| < \varepsilon.$$

Corollary: There exist subspaces X and Y and a product φ of projections onto X, Y and E so that

$$|\varphi(X, Y, E)(u) - v| < \varepsilon.$$

Iterates of 1 point (and of points near to it) diverge

Let H be an infinite dimensional Hilbert space.

There exist 2 closed and convex sets $C, D \subset \ell_2$ with $0 \in C \cap D$, and a sequence of iterates of nearest point projections of a point z on these sets which does not converge in norm, since it approximately contains an ON sequence.
[Hein Hundal 2004]

There exist 5 closed subspaces $E, X_{even}, X_{odd}, Y_{even}, Y_{odd}$ of H and a sequence of iterates of orthonormal projections of a point z on these spaces which does not converge in norm, since it approximately contains an ON sequence.
[Adam Paszkiewicz, 2012]

There exist 3 closed subspaces E, X, Y of H and a sequence of iterates of orthonormal projections of a point z on these spaces which does not converge in norm, since it approximately contains an ON sequence.
[Eva Kopecká & Vladimír Müller, 2013]

Iterates of ALL points diverge

Let H be an infinite dimensional Hilbert space.

There exist 3 closed subspaces X_1, X_2, X_3 with the following property. For every $0 \neq w_0 \in H$ there is a sequence $k_1, k_2, \dots \in \{1, 2, 3\}$ so that the sequence of iterates defined by $w_n = P_{X_{k_n}} w_{n-1}$ does not converge in norm.

When projecting on 5 closed subspaces this can be achieved using just 2 fixed sequences of indices:

There exists a sequence $k_1, k_2, \dots \in \{1, 2, 3\}$ with the following property. Every infinite dimensional Hilbert space H has closed subspaces X, Y, X_1, X_2, X_3 so that if $0 \neq z \in H$, and $u_0 = P_X z$, $v_0 = P_X P_Y z$, then at least one of the sequences of iterates $\{u_n\}_{n=1}^{\infty}$ or $\{v_n\}_{n=1}^{\infty}$ defined by $u_n = P_{X_{k_n}} u_{n-1}$, $v_n = P_{X_{k_n}} v_{n-1}$ does not converge in norm.

[Kopecká & Paszkiewicz, 2015]

3 subspaces: from 1 bad point to all points bad

Let H be an infinite dimensional Hilbert space, and $X_1, X_2, X_3 \subset H$ be 3 of its closed subspaces.
Let Z be the set of good points in H, that is of all $z_0 \in H$ so that for every sequence $j_1, j_2, \dots \in \{1, 2, 3\}$ the sequence defined by $z_n = P_{X_{j_n}} z_{n-1}$ does converge in norm.
Then Z is a closed subspace.

ASSUME $Z \neq H$

Then $L = Z^{\perp}$ is an infinite dimensional subspace of H, and in L all points are bad w.r.t. the spaces $\tilde{X}_i = L \cap X_i$, $i = 1, 2, 3$, that is:

For every $0 \neq w_0 \in L$ there is a sequence $k_1, k_2, \dots \in \{1, 2, 3\}$ so that the sequence of iterates defined by $w_n = P_{\tilde{X}_{k_n}} w_{n-1}$ does not converge in norm.

5 subspaces: from 1 bad point to all points bad

Fix $k_1, k_2, \dots \in \{1,2,3\}$ with the following property:
Every infinite dimensional (separable) Hilbert space H^m contains a point $z_0^m \in H$ and closed subspaces X_1^m, X_2^m, X_3^m so that the sequence $z_n^m = P_{X_{k_n}} z_{n-1}^m$ diverges.
DEFINE:

$$
\begin{aligned}
H &= H^1 \oplus_2 H^2 \oplus_2 H^3 \oplus \dots \\
X_i &= X_i^1 \oplus_2 X_i^2 \oplus_2 X_i^3 \oplus \dots \quad i = 1,2,3 \\
X &= \overline{\mathrm{span}}\{z^1, z^2, z^3, \dots\} \\
Y &= (X + X^{\perp})/2.
\end{aligned}
$$

THEN:

If $0 \neq z \in H$, and $u_0 = P_X z$, $v_0 = P_X P_Y z$, then at least one of the sequences of iterates $\{u_n\}_{n=1}^{\infty}$ or $\{v_n\}_{n=1}^{\infty}$ defined by $u_n = P_{X_{k_n}} u_{n-1}$, $v_n = P_{X_{k_n}} v_{n-1}$ diverges.

Projections on 3 subspaces do not have to converge

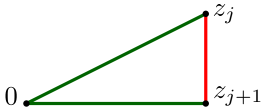
Let H be an infinite dimensional Hilbert space.
There exist 3 closed subspaces X_1, X_2, X_3 with the following property:
For every $0 \neq w_0 \in H$ there is a sequence $k_1, k_2, \cdots \in \{1, 2, 3\}$ so that the sequence of iterates defined by $w_n = P_{X_{k_n}} w_{n-1}$ converges weakly but does not converge in norm.

Simple condition implying convergence

$L_1, L_2, \ldots, L_K$ closed subspaces of a Hilbert space H
$z_n = P_{k_n} z_{n-1}$ iterates of orthoprojections of a point z

Lemma

The sequence $\{|z_n|\}$ is decreasing, hence convergent.



Lemma

Suppose there is $c > 0$, so that $|z_j - z_k|^2 \leq c(|z_j|^2 - |z_k|^2)$, for all $j \leq k$. Then $\{z_n\}$ converges in norm.

Proof.

If $j < k$ are large, then $|z_j - z_k|^2 \leq c(|z_j|^2 - |z_k|^2) < \varepsilon$
since $\{|z_n|\}$ is Cauchy.
$\Rightarrow \{z_n\}$ is Cauchy $\Rightarrow \{z_n\}$ converges $\square$

Finite (co)-dimension $\Rightarrow$ convergence

$L_1, L_2, \ldots, L_K$ closed subspaces of a Hilbert space H
of finite dimension or codimension
$z_n = P_{k_n} z_{n-1}$ iterations of orthoprojections of a point z

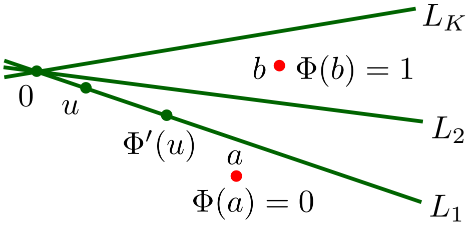
Theorem

For all $j \leq k$,

$$|z_j - z_k|^2 \leq c(|z_j|^2 - |z_k|^2),$$

where the constant $c = c(K, d) > 0$ depends on the number K of the spaces and their maximal dimension or codimension d (for each space we choose the one which is finite) only. Consequently, the sequence $\{z_n\}$ converges in norm.

Smooth separation theorem



if $a = 0$ then $\Phi(x) = |x|^2$ works
since $\Phi'(x) = 2x$
!BUT!
$|a| \approx |b| \approx 1$
in a typical application

Theorem (Kirchheim, Kopecká, Stefan Müller, 2011)

Let $L_1, L_2, \ldots, L_K$ be subspaces of $\mathbb{R}^d$ and let $a, b \in \mathbb{R}^d$ be two points. There exists a differentiable function $\Phi : \mathbb{R}^d \to \mathbb{R}$, so that

(i) $\Phi(b) - \Phi(a) = |b - a|^2$;

(ii) $\Phi'(L_i) \subset L_i$ *for* $i = 1, \ldots, K$;

(iii) *the mapping* $\Phi' : \mathbb{R}^d \to \mathbb{R}^d$ *is Lipschitz with a constant* c *depending on* K *and* d *only.*

Proof: involved application of Whitney's theorem on extending of functions and their derivatives.

Corollary

Let $w = b - a$ and $F : \mathbb{R}^d \to \mathbb{R}$ be the function defined by $F(x) = \langle w, x \rangle - \Phi(x)$ for $x \in \mathbb{R}^d$. Then $F(b) - F(a) = 0$, $F' = w - \Phi'$, and if $i \in \{1, \ldots, K\}$, then

$$\langle w, v \rangle \leq \langle F'(x), v \rangle + c \operatorname{dist}(x, L_i),$$

for any $x \in \mathbb{R}^d$ and v orthogonal to L_i, with $|v| = 1$.

Proof.

For a given i, let $\tilde{x}$ be the orthogonal projection of x onto L_i. Then $\langle \Phi'(\tilde{x}), v \rangle = 0$ and since Φ' is c-Lipschitz,

$$\begin{aligned}|\langle w - F'(x), v \rangle| = |\langle \Phi'(x), v \rangle| = |\langle \Phi'(x) - \Phi'(\tilde{x}), v \rangle| \leq |\Phi'(x) - \Phi'(\tilde{x})| \\ \leq c|x - \tilde{x}| = c \operatorname{dist}(x, L_i).\end{aligned}$$

$\square$

Smooth separation $\Rightarrow$ projections converge

Let $L_1, L_2, \dots, L_K$ be closed subspaces of a Hilbert space X. Suppose for every $a, b \in X$ there exists a differentiable function $\Phi : X \to \mathbb{R}$, so that

(i) $\Phi(b) - \Phi(a) = |b - a|^2$;

(ii) if $x \in L_i$ then $\Phi'(x) \in L_i$;

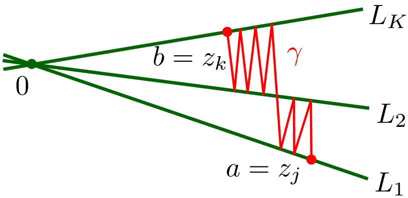
(iii) the mapping $\Phi' : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz with a constant c.

Then for every sequence $\{z_n\}$ of projections on the sets $L_1, \dots, L_K$

$$|z_j - z_k|^2 \le c(|z_j|^2 - |z_k|^2),$$

if $j < k$. Consequently, the sequence $\{z_n\}$ converges in norm.

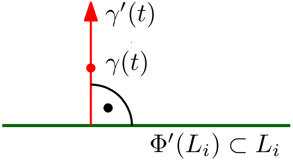
Heuristics: smooth separation $\to$ rate of convergence



curve $\gamma : [0, s] \to \mathbb{R}^d$
connects via the iterates z_n
$\gamma(0) = a = z_j$ with
$\gamma(s) = b = z_k$
γ replaces the direct connection
$w = b - a$

$$
\begin{aligned}
|z_j - z_k|^2 &= \langle w, \gamma(s) - \gamma(0) \rangle \\
&= \int_0^s \langle w, \gamma'(t) \rangle \, dt \\
&\approx \int_0^s \langle w - \Phi'(\gamma(t)), \gamma'(t) \rangle \, dt \\
&= \langle w, b \rangle - \Phi(b) - (\langle w, a \rangle - \Phi(a)) \\
&= \langle w, b - a \rangle - |b - a|^2 = 0
\end{aligned}
$$

$$\langle \Phi'(\gamma(t)), \gamma'(t) \rangle \approx 0$$

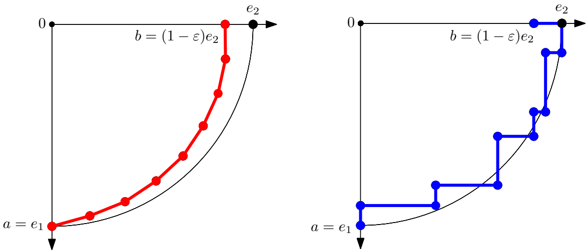


(We didn't mention the error of order $c \sum |z_{i+1} - z_i|^2 = c(|z_j|^2 - |z_k|^2)$.)

smooth separation $\to$ rate of convergence

$$
\begin{aligned}
|z_j - z_m|^2 &= \langle w, b-a\rangle = \langle w, \gamma(s) - \gamma(0)\rangle = \int_0^s \langle w, \gamma'(t)\rangle \, dt \\
&= \sum_{i=j}^{m-1} \int_{s_i}^{s_{i+1}} \langle w, v_i\rangle dt \\
&\le \sum_{i=j}^{m-1} \int_{s_i}^{s_{i+1}} \langle F'(\gamma(t)), \gamma'(t)\rangle dt + c \sum_{i=j}^{m-1} \int_{s_i}^{s_{i+1}} s_{i+1} - t \, dt \\
&= \sum_{i=j}^{m-1} F(z_{i+1}) - F(z_i) + c \sum_{i=j}^{m-1} \int_0^{s_{i+1}-s_i} t \, dt \\
&= F(z_m) - F(z_1) + c/2 \sum_{i=j}^{m-1} (s_{i+1} - s_i)^2 = C \sum_{i=j}^{m-1} |z_{i+1} - z_i|^2 \\
&= C(|z_j|^2 - |z_m|^2)
\end{aligned}
$$

Monotone curves with only few different derivatives do *not* connect distant points of the sphere.



Assume $\gamma : [0, s] \to \ell_2$ is an absolutely continuous curve with endpoints a and b so that

(i) the distance $|\gamma(t)|$ from the origin is a decreasing function of t on $[0, s]$, and

(ii) $\gamma'(t)$ takes on at most K different values for almost all $t \in [0, s]$.

Then $|a - b|^2 \le c(|a|^2 - |b|^2)$.

$c = c(K) > 0$ depends only on the number K of the different derivatives of γ.