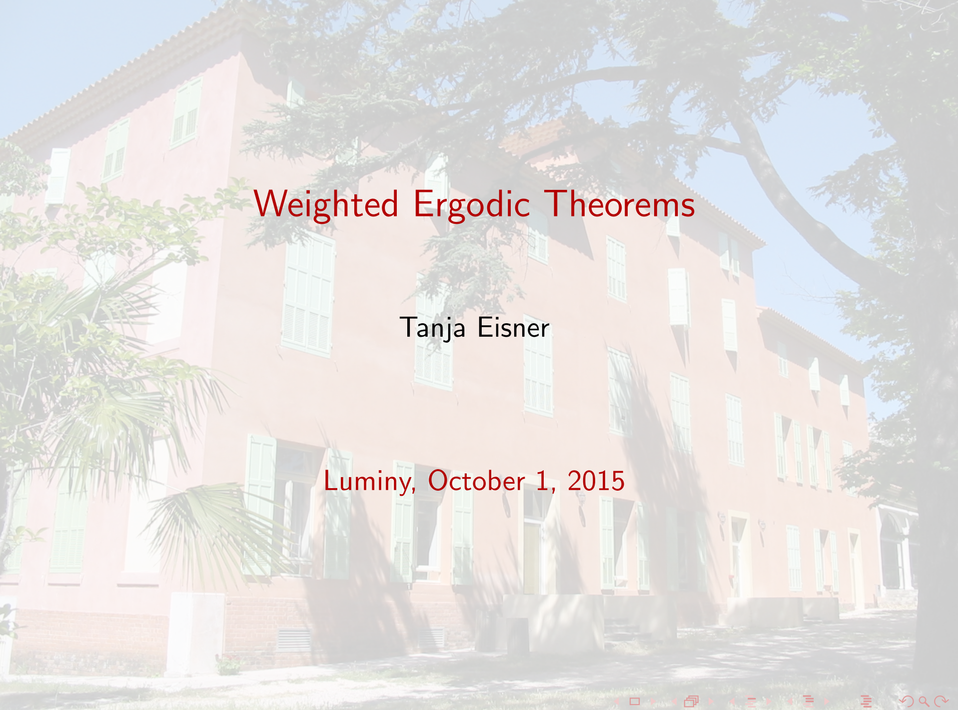

Weighted Ergodic Theorems

Tanja Eisner

Luminy, October 1, 2015

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$$T \text{ ergodic} \iff \mathrm{Fix}\, T = \{\mathbf{1}\}$$

Pointwise Ergodic Theorem, Birkhoff, 1931

For every $f \in L^1(X, \mu)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f(x) = \int_X f d\mu$$

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"time mean = space mean"

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Find "good" bounded weights $(a_n) \subset \mathbb{C}$:

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Pointwise conv.: no characterization

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We say: (λ^n), $\lambda \in \mathbb{T}$, is a family of WW-weights

WW Theorem: Proof

Idea: Use

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hence uniform conv. to 0 for weakly mixing fcts.

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So we have:

$$\limsup_{N\to\infty} \sup_{\lambda\in\mathbb{T}} \left| \frac{1}{N} \sum_{n=1}^{N} \lambda^n T^n f(x) \right| \le \|f\|_{U^2}$$

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Bourgain's Return Times Theorem '89: $\forall (X, \mu, T)\ \forall f \in L^1(X, \mu)$, $(f(T^n x))$ is a good weight for a.e. x.

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sup over $(g(S^n y))$ with $\|g\|_{W^{d(2^l-1),2^l}} \leq 1$

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Boshernitzan '94: Not all $e(p(n))$ are Cesàro summable (growth condition)

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So: $(e(p(n)))$ good (WW-)weight $\iff$ it is so for nilsystems.

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  - General case open

One More Example

$$\text{Möbius fct } \mu(n) = \begin{cases} 1, & n = p_1 \dots p_{2k},\ p_j \text{ distinct}, \\ -1, & n = p_1 \dots p_{2k+1},\ p_j \text{ distinct}, \\ 0, & n \text{ not square-free.} \end{cases}$$

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Sarnak's conjecture '11:

$\forall\ (X, T)$ with entropy 0, $\forall f \in C(X)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n)(T^n f)(x) = 0$$

$\forall x \in X$.

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Möbius fct $\mu(n) = \begin{cases} 1, & n = p_1 \dots p_{2k},\ p_j \text{ distinct}, \\ -1, & n = p_1 \dots p_{2k+1},\ p_j \text{ distinct}, \\ 0, & n \text{ not square-free.} \end{cases}$

$(\mu(n))$ is a good weight (Sarnak '11, El Abdalaoui, Kulaga, Lemańczyk, de la Rue '14, T.E. '15)

Sarnak's conjecture '11:

$\forall\ (X, T)$ with entropy 0, $\forall f \in C(X)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n)(T^n f)(x) = 0$$

$\forall x \in X$. Or:

$\mu \perp$ deterministic sequences

MANY THANKS

FOR YOUR ATTENTION