Eigenvalues of minimal Cantor systems

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Joint works with :

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- Alejandro MAASS : CDHM 03, BDM 05, BDM 10, DFM 14, DFM 15
- Bernard HOST : CDHM 03
- Xavier BRESSAUD : BDM 05, BDM 10
- Samuel PETITE : CDP 14
- Alexander FRANK : DFM 14, DFM 15

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When we do not have the equality, can we precise those eigenvalues in Eig_μ that are in Eig ?

Context : minimal Cantor systems

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(X, T, μ) is ergodic

Well-known results

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- Dekking 78 : For primitive substitutions of constant length p :

$$Eig = Eig_\mu = \{a/qp^n | a \in \mathbb{Z}\}$$

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- Downarowicz-Lacroix 96 and Iwanik 96 : There exist Toeplitz subshifts with $Eig \neq Eig_{\mu}$ (for some ergodic measures)
- Indeed, any countable subgroup of $[0, 1]$ containing infinitely many rationals can be realized as Eig_{μ} of some Toeplitz subshift

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- Numeration systems for return times

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Thus $f(x) = \lambda^{r(x)}$ "almost" satisfies $f \circ T = \lambda f(x)$.

We need good sequences of partitions

Kakutani-Rohlin partitions :

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Herman-Putnam-Skau 92

Incidence matrices

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$$H(n) = M(n)H(n-1)(H(1) = M(1))$$

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- Toeplitz subshifts : $H(n) = p_n(1, \ldots, 1)^t$ (Gjerde-Johansen 2000)

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Theorem. Let μ be an invariant measure of (X, T). $\lambda = \exp(2i\pi\alpha) \in Eig_\mu(X, T)$ if and only if there exist real functions $\rho_n : \{1, ..., C(n)\} \to \mathbb{R}$, $n \in \mathbb{N}$, such that

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Idea of the proof (classical) : Consider $\mathbb{E}_\mu(\lambda^{r_n} | \mathcal{P}(n))$

A NSC to be a continuous eigenvalue

Theorem. (DFM 2015) λ is a continuous eigenvalue of (X, T) if and only if

$$\sum_n \max_{x \in X} |||\langle s_n(x), \alpha H_n \rangle||| < \infty.$$

Some properties of continuous eigenvalues

Proposition. (Itza-Ortiz 07 and CDHM 03) For all invariant measure μ, *Eig* is a subgroup of the group G spanned by $\{\mu(U) | U \text{ clopen set }\}$:

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Thus, the only realizable eigenvalue subgroups are $\mathbb{Z}$ and $\mathbb{Z} + \alpha\mathbb{Z}$.

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Corollary.

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- (BDM 05) If

$$\sum_{m\geq 1} \left(\frac{\sup_{k\in\{1,\ldots,C(m+1)\}} h_k(m+1)}{\inf_{k\in\{1,\ldots,C(m)\}} h_k(m)} \right) \sup_{k\in\{1,\ldots,C(m)\}} \mid \lambda^{h_k(m)} - 1 \mid < \infty$$

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(1) $\Rightarrow f_n = \mathbb{E}(f|\mathcal{P}(n))$, then with Martingale Theorem

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(1) $\Leftarrow$

Lemma. The sequence of random variables $(\tau_n; n \in \mathbb{N})$ is a non-stationary Markov chain. ($\tau_n(x) =$ name of the towers including x in partition $\mathcal{P}(n)$)

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Lemma. There exist $c \in \mathbb{R}_+$ and $\beta \in [0,1[$ such that for all $n, k \in \mathbb{N}$, with $k \leq n$,

$$\sup_{1 \leq t \leq C(n-k), 1 \leq \bar{t} \leq C(n)} |\mu[\tau_n = \bar{t} | \tau_{n-k} = t] - \mu[\tau_n = \bar{t}]| \leq c\beta^k \ .$$

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$$g_n(x) = \sum_{j=1}^{n-1} < s_j(x), P(j)v > \ ,$$

Lemma. $(f_n = g_n - \mathbb{E}_\mu(g_n); n \geq 1)$ converges in $L^2(X, \mathcal{B}_X, \mu)$.

Ideas of proof

Lemma. There exist $c \in \mathbb{R}_+$ and $\beta \in [0,1[$ such that for all $n, k \in \mathbb{N}$, with $k \leq n$,

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Using the following decomposition

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$$Y_n = \mathbb{E}_\mu(X_n | \mathcal{P}(n)) \text{ and } Z_n =< s_n, P(n)v > - \mathbb{E}_\mu(< s_n, P(n)v > | \mathcal{P}(n)) \ .$$

Ideas of proof

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$$||f_n - f_{n-1}||_\infty \le L \max_{1\le k\le C(n-1)} |\lambda^{h_k(n-1)} - 1| \ .$$

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converges uniformly.

Examples

(X, T, μ) linearly recurrent with $M(n)$, $n \geq 2$ in

$$\left\{ A = \begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}. \right\}$$

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For $\limsup a_n/n = \delta$ there are (LR) examples with non trivial eigenvalues, none of them being continuous.

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Let (X, T) be a Toeplitz subshift.
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Theorem. (BDM10)
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$$\frac{q}{(q, p_n)} \leq d.$$

An example

$$\sigma_n(1) = (12)^{t_n}222, \quad \sigma_n(1) = 1(12)^{t_n}12, \quad 2t_n + 3 = 3^k$$

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2. If μ and ν are different ergodic measures then $I_\mu \cap I_\nu = \emptyset$.

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Proposition (BDM10). For rank d minimal Cantor systems.
$\# \mathcal{M}_{erg}(X, T) + \max$ number of $\mathbb{Q}-$independent cont.eig. $\leq d + 1$.