Distributional limits of positive, ergodic stationary processes

& infinite ergodic transformations

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Frontiers in Operators Dynamics
CIRM

29th Sep 2015

work in progress w. Benjamin Weiss

1. Stationary, stochastic processes

- `stationary process` (SP): $(\Omega, \mathcal{A}, P, S, \varphi)$ where $(\Omega, \mathcal{A}, P, S)$ is a PPT & $\varphi : \Omega \to \mathbb{R}$ measurable.

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- *positive* if $\varphi \geq 0$;

- *ergodic* (ESP) if $(\Omega, \mathcal{A}, P, S)$ is ergodic.

- *independent* (identically distributed) (IIDSP) if $\{\varphi \circ S^n : \ n \geq 0\}$ are independent random variables.

2. Distributional convergence

For (X, d) a metric space set

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For $Y_n,\ Y \in \mathtt{RV}(X)$ say that $Y_n \xrightarrow[n\to\infty]{\mathtt{dist}} Y$ if

$$E(g(Y_n)) \xrightarrow[n\to\infty]{} E(g(Y))\ \forall\, g \in C_B(X),\ \mathtt{i.e.}\ \mathtt{dist}\, Y_n \xrightarrow[n\to\infty]{\mathcal{P}(X)} \mathtt{dist}\, Y.$$

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d-Vasershtein distance: $\mathfrak{v} = \mathfrak{v}_d$ on $\mathtt{RV}(X)$:

$\mathfrak{v}(Y_1, Y_2) :=$

$$\min\{E(d(Z_1, Z_2)) :\ Z = (Z_1, Z_2) \in \mathtt{RV}(\mathbb{R}_+ \times \mathbb{R}_+),\ Z_i \overset{\mathtt{dist}}{=} Y_i\ (i = 1, 2)\}.$$

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In case (X, d) is compact, then for $Y_n,\ Y \in \mathtt{RV}(\mathbb{R}_+)$,

$$Y_n \xrightarrow[n\to\infty]{\mathtt{dist}} Y \iff \mathfrak{v}(Y_n, Y) \xrightarrow[n\to\infty]{} 0.$$

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(SLT) $$\frac{\varphi_n}{b(n)} \xrightarrow[n\to\infty]{\texttt{dist}} Z_\gamma$$ where Z_γ is γ-stable i.e.

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Note that Z_1 is constant.

3. 1-regularly varying normalizing constants

A. & Omri Sarig, [ETDS, 2014] $\exists$ +-ive ESP $(\Omega, \mathcal{F}, P, R, \varphi)$ so that

$$\frac{1}{b(n)} \sum_{k=0}^{n-1} \varphi \circ R^k \xrightarrow[n\to\infty]{\texttt{dist}} e^{\frac{1}{2}\mathcal{N}(0,1)^2}$$

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R is aka the "Pascal-adic" or "de Finetti" transformation.

3. The general positive distributional limit

[🚲, 2015?] A. & Benjamin Weiss: Let $Y \in \mathrm{RV}(\mathbb{R}_+)$, then $\exists$

- an odometer $(\Omega, \mathcal{F}, P, S)$,
- an increasing 1 reg. var. function $b : \mathbb{R}_+ \to \mathbb{R}_+$ and
- a positive measurable function $\varphi : \Omega \to \mathbb{R}_+$ so that

$$\frac{1}{b(n)} \sum_{k=0}^{n-1} \varphi \circ S^k \xrightarrow[n\to\infty]{\mathtt{dist}} Y.$$

☺ [🚲, 2015?] Such a function exists on any EPPT.

4. Strong distributional convergence

For $(X, \mathcal{B}, m)$ a measure space, $F_n : X \to [0, \infty]$ measurable, $P \in \mathcal{P}(X, \mathcal{B})$, $P \ll m$ and $Y \in \mathtt{RV}([0, \infty]) := \{\mathtt{random\ variables\ on}\ [0, \infty]\}$:

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¶[Eagleson] If $(X, \mathcal{B}, m, T, f)$ is an $\mathtt{ESP}$, $a(n) \to \infty$ &

$$\frac{1}{a(n)} \sum_{k=0}^{n-1} f \circ T^k \xrightarrow[n\to\infty]{P\text{-}\mathtt{dist}} Y \text{ for some } P \in \mathcal{P}(X, \mathcal{B})\ P \ll m,$$

then

$$\frac{1}{a(n)} \sum_{k=0}^{n-1} f \circ T^k \xrightarrow[n\to\infty]{\mathfrak{d}} Y.$$

5. The general case for [🚲]

Fix $Y \in \mathtt{RV}(\mathbb{R}_+)$. As above there is an odometer $(Y, \mathcal{C}, \nu, \tau)$ and a function $F : Y \to \mathbb{R}_+$ satisfying [🚲].

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If $K_n(x) = \#\{k \le n : \ S(x) \in A\}$ then $K_n(x) \sim m(A)n$ and $S_n(f)^{(T)}(f) = S_{K_n}^{(T_A)}(F \circ \Pi)$.

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For $\epsilon > 0$, $N_\pm := (1 \pm \epsilon)m(A)n$, for large n with high probability,

$$(1-\epsilon)\frac{S_{N_-}^{(T_A)}(F \circ \Pi)}{b(N_-)} \lesssim \frac{S_n^{(T)}(f)}{b(n)} \lesssim (1+\epsilon)\frac{S_{N_+}^{(T_A)}(F \circ \Pi)}{b(N_+)}.$$

6. Castles

Castle $\mathfrak{W} = \{W_j : 1 \le j \le k\}$ where the $W_j = (I_{1,j}, I_{2,j}, \ldots, I_{h,j})$ are columns of intervals with equal widths and heights $h =: |\mathfrak{W}|$.

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Castle $\mathfrak{W}$ equipped with partial transformation

$$T_{\mathfrak{W}} : U(\mathfrak{W}) \cong (0,1) \to U(\mathfrak{W})$$

defined by the translations $I_{k,j} \mapsto I_{k+1 \mod h,j}$.

7. Refinements & inverse limit $\mathtt{PPT}$s

The castle $\mathfrak{W}' = \{W'_j : 1 \le j \le k'\}$, **refines** $\mathfrak{W}$ $\quad (\mathfrak{W}' \succ \mathfrak{W})$ if

$$U(\mathfrak{W}') = U(\mathfrak{W}),\ \mathtt{Top}(\mathfrak{W}') \subset \mathtt{Top}(\mathfrak{W})\ \&\ \ T_{\mathfrak{W}'}|_{\mathfrak{W} \setminus \mathtt{Top}(\mathfrak{W})} \equiv T_{\mathfrak{W}}.$$

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The refinement $\mathfrak{W}' \succ \mathfrak{W}$ is **transitive** if

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☺ (1) If $\dots \prec \mathfrak{W}_n \prec \mathfrak{W}_{n+1} \prec \dots\ \&\ m(\mathtt{Top}(\mathfrak{W}_n)) \xrightarrow[n\to\infty]{} 0$,
then $\exists$ PPT $\ \tau := \varprojlim_{n\to\infty} \mathfrak{W}_n$ of $(0,1) = U(\mathfrak{W}_n)$ so that for m-a.e. $x \in (0,1)$, $\ T_{\mathfrak{W}_n}(x) = \tau(x)\ \ \forall\ n$ large.

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☺ (1) If $\dots \prec \mathfrak{W}_n \prec \mathfrak{W}_{n+1} \prec \dots$ & $m(\mathtt{Top}(\mathfrak{W}_n)) \xrightarrow[n\to\infty]{} 0$, then $\exists$ PPT $\tau := \varprojlim_{n\to\infty} \mathfrak{W}_n$ of $(0,1) = U(\mathfrak{W}_n)$ so that for m-a.e. $x \in (0,1)$, $T_{\mathfrak{W}_n}(x) = \tau(x)$ $\forall$ n large.

☺ (2) If in addition, infinitely many of the refinements $\mathfrak{W}_n \prec \mathfrak{W}_{n+1}$ are transitive then $\varprojlim_{n\to\infty} \mathfrak{W}_n$ is ergodic.

7. Rokhlin towers and odometers

Homogeneous refinement: $\mathfrak{W}' \succ \mathfrak{W}$ where

$$W_j' =^{k'} \widetilde{W} \quad \text{where } \widetilde{W} := \bigodot_{q=1}^{Q} (W_{\kappa_q})^{\circledast s_q}.$$

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If $\mathfrak{W}_n \prec \mathfrak{W}_{n+1} \prec \dots$ are homogeneous refinements, then

$$\varprojlim_{n\to\infty} \mathfrak{W}_n = \varprojlim_{n\to\infty} \widetilde{W_n} \quad \text{is an odometer.}$$

8. Step functions & symmetric random variables

`Step function on a castle` $\mathfrak{W}$: a function $F : \mathfrak{W} \to \mathbb{R}_+$ which is constant on each interval of $\mathfrak{W}$.

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$(\Omega, \mathfrak{y})$-distributed castle process: $(\mathfrak{W}, F)$ where

$$\mathfrak{W} = \{W_\omega = (I_{1,\omega}, \ldots, I_{h,\omega}) : \ \omega \in \Omega\} \ \ \& \ \exists \ c = c(\mathfrak{W}, F) \text{ such that}$$

$$E(W_\omega) := \frac{1}{h} \sum_{k=1}^{h} F(I_{k,\omega}) = c\mathfrak{y}(\omega).$$

9. The ESP constructed for symmetric Y

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$$\mathfrak{v}\left(\frac{S_k(F_{n+1})}{k\gamma(k)}, \mathfrak{y}\right) \quad < \quad \Delta_{n+1}.$$

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(iii) for each $k > \Delta_{n+1}$ and $\omega \in \Omega$,

$$P(S_k(F_{n+1}) = \mathfrak{y}(\omega) k\gamma(k)(1 \pm \Delta_{n+1}) \| W_\omega^{(n+1)}) > 1 - \Delta_{n+1}.$$

10. The function constructed from blocks

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Recursions for the block arrays: of form

$$w_\ell^{(n+1)} = \left(\bigodot_{q=1}^{Q_n} (w_{\kappa_q}^{(n)})^{\odot s_q} + \mathcal{E}_{q,\ell}^{(n)}] \right)^{\odot s_{Q_n}} + \mathcal{D}_\ell^{(n)}$$

where $\odot$ means concatenation and $\mathcal{E}_{q,\ell}^{(n)} \in \mathbb{R}_+^{s_q h_n}$ $(1 \le q \le Q_n)$ are "small modifications".

11. Block statistics & ϵ-normalizations

For $w \in \mathbb{R}_+^h$ a block,

$$M(w) := \max_{1 \le j \le h} w_j, \quad \Sigma(w) := \sum_{1 \le j \le h} w_j \quad \& \quad E(w) := \frac{\Sigma(w)}{|w|}.$$

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Block $w \in \mathbb{R}_+^h$ is ϵ-`normalized` if

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If w is a block & $\epsilon > 0$, then $w^{\odot m}$ is ϵ- normalized $\forall$ large m.

12. Basic Lemma for

For: $0 < \Delta < 1$, $w \in \mathbb{R}_+^h$ Δ-normalized,
$0 < \kappa \le \Delta E(w)$, $\mathfrak{d} > 0$, $q > \frac{1}{\Delta}$ & for $\mu \in \mathbb{N}$ large enough: if
$m := \mu q$ &

$$w' = w^{(\mu)} := w^{\odot m} + \kappa q h 1_{[1,mh] \cap qh\mathbb{Z}},$$

then

(i) w' is $\mathfrak{d}$-normalized;

(ii) $E(w') = E(w) + \kappa$;

(iii) $P(S_k(w') = S_k(w^{\odot m}) \ \ \forall\ 1 \le k \le \sqrt{\Delta} qh) \ge 1 - \sqrt{\Delta}$;

(iv) $S_k(w') = kE(w)(1 \pm 2\sqrt{\Delta}) \ \forall\ \sqrt{\Delta} qh \le k \le qh$;

(v) $S_k(w') = k(E(w) + \kappa)(1 \pm (\Delta \wedge \frac{1}{k} + \frac{\Delta qh}{k})) \ \forall\ k > qh$.

13. Block average changes

- $0 < \Delta < 1, h \in \mathbb{N}$ & $\mathcal{F} \subset \mathbb{R}_+^h$ a Δ-normalized h-block array.
- For $J \subset (1, \infty)$ finite, $\forall \ \ \beta > 0$ & $\mathcal{E} > 0$, and $Q \in \mathbb{N}$ large enough, there exist
- an $\mathcal{E}$-normalized, Qh–block array

$$\{v(w,t): \ w \in \mathcal{F}, \ t \in J\} \subset \mathbb{R}_+^{Qh}$$

so that for each $w \in \mathcal{F}$ & $t \in J$,

- $v(w,t)$ Δ-`approximates` w in the sense that

$$P(S_k(v(w,t)) = S_k(w^{\odot Q}) \ \ \forall \ 1 \le k \le \Delta h) > 1 - 2\Delta.$$

and

$$E(v(w,t)) = tE(w).$$

14. Simultaneous normalization of block partial sums

Let $0 < \Delta < 1$, $h \in \mathbb{N}$ and let $\mathcal{F} \subset \mathbb{R}_+^h$ be a Δ-normalized h-block array then

$$\forall \quad \Gamma > 1,\ \beta > 0,\ \mathcal{E} > 0 \quad \& \quad Q \in \mathbb{N} \ \text{ large enough,}$$

there exists a $\mathcal{E}$-normalized, Qh–block array

$$\mathcal{V} = \{v(w) : \ w \in \mathcal{F}\} \subset \mathbb{R}_+^{Qh}$$

so that so that for each $w \in \mathcal{F}$,

(i) $\qquad E(v(w)) = \Gamma E(w);$

(ii) $\qquad P(S_k(v(w)) = S_k(w^{\odot Q}) \ \ \forall\ 1 \le k \le \Delta h) > 1 - 2\Delta.$

Moreover there are constants $\gamma(k) > 0,\ (\Delta h \le k \le Qh)$ so that

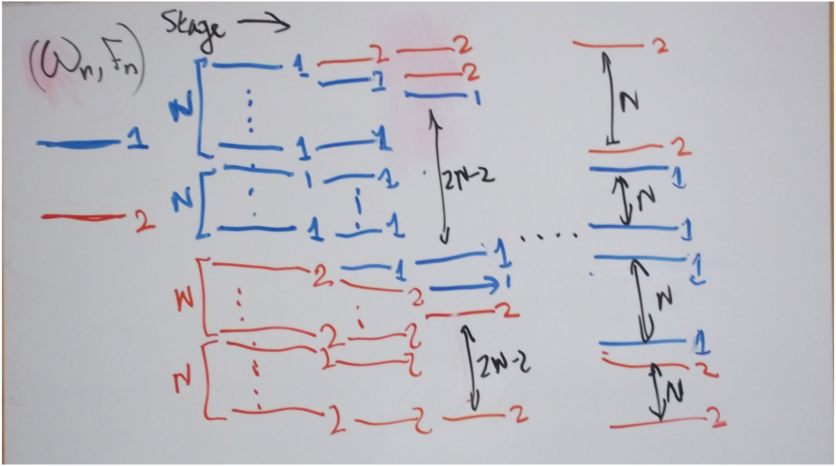
(iii) $\qquad 1 = \gamma([\Delta h]) \le \gamma([\Delta h] + 1) \le \dots \ \le \gamma(Qh) = \Gamma;$

(iv) $\qquad 0 \le \gamma(k+1) - \gamma(k) \le \beta;$

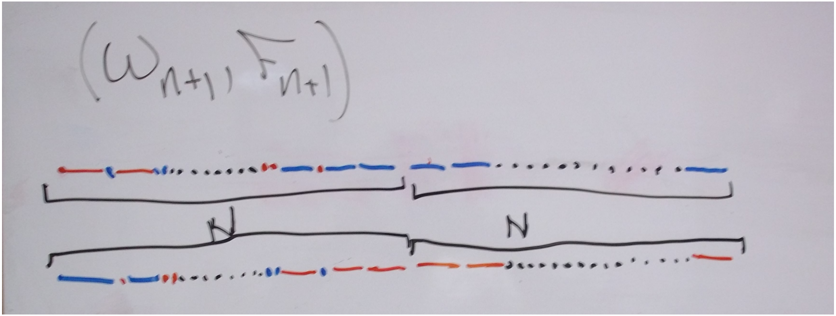
and such that for each $w \in \mathcal{F}$,

(iii) $\quad P([S_k(v(w)) = k\gamma(k)E(w)(1 \pm \mathcal{E})]) \ge 1 - \mathcal{E} \quad \forall\ k > \mathcal{E}Qh.$

15. Inductive stage in for $Y = 1, 2$ w.p. $\frac{1}{2}$: Intermediates



16. Inductive stage in 🚜 for $Y = 1, 2$ w.p. $\frac{1}{2}$: Next



17. Consequences for infinite ergodic theory

For each $Y \in \mathtt{RV}((0,\infty))$ there is a $\mathtt{CEMPT}$ $(X, \mathcal{B}, m, T)$: and a 1-regularly varying function $a(n)$ so that

$$\frac{S_n^{(T)}(f)}{a(n)} \xrightarrow{\mathfrak{d}} Y \int_X f dm \quad \forall\ f \in L^1(m)_+.$$

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Proof By [] $\exists$ an $\mathtt{ESP}$ $(\Omega, \mathcal{F}, P, S, \varphi)$ and a 1-regularly varying function $a(n)$ so that

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Define $(X, \mathcal{B}, m, T)$ by

$X := \{(\omega, n) :\ \omega \in \Omega,\ 1 \le n \le \varphi(\omega)\},\ m(A \times \{n\}) = P(A \cap [\varphi \ge n])$ &

$$T(x,n) := \begin{cases} (Sx, 1) & n = \varphi(x), \\ (x, n+1) & n < \varphi(x). \end{cases}$$

18. Conclusion of proof by inversion

Let $a := b^{-1}$, then a is 1-regularly varying.

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$$S_n(1_\Omega) > K \quad \Longleftrightarrow \quad \varphi_K < n.$$

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Proof

$$S_n(1_\Omega) > K \iff \varphi_K < n.$$

$\therefore$ for $t > 0$, $P(Y = t) = 0$, $n, N \geq 1$ & $N \sim ta(n)$,

$$\begin{aligned} P(S_n(1_\Omega) > ta(n)) &\approx P(S_n(1_\Omega) > N) \\ &= P(\varphi_N < n) \\ &\approx P(\varphi_N < \frac{b(N)}{t}) \\ &\xrightarrow[n\to\infty]{} P(\frac{1}{Y} < \frac{1}{t}) \\ &= P(Y > t). \quad \boxtimes \end{aligned}$$

19. The End

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Thank you for listening.