$\mathcal{B}$-free dynamics - a view through the window

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C.I.R.M. Luminy, 7.12.2016

The background to this work

- A new approach to cut-and-project schemes (weak model sets) (Keller, Richard 2015, $\to$ E.T.& D.S.)
- $\mathcal{B}$-free systems as special case (Square-free numbers: Baake, ...?)
- $\mathcal{B}$-free sets and dynamics (Bartnicka, Kasjan, Kułaga-Przymus, Lemańczyk 2015, $\to$ T.A.M.S.)

The formal framework

- $\mathcal{B} \subseteq \mathbb{N}_0$ **primitive**
- $\widetilde{H} := \prod_{b \in \mathcal{B}} \mathbb{Z}/b\mathbb{Z}$ compact, metrizable abelian group.
- $\Delta : \mathbb{Z} \to \widetilde{H}$, $(\Delta(n))_b = n \mod b$ (group monomorphism).
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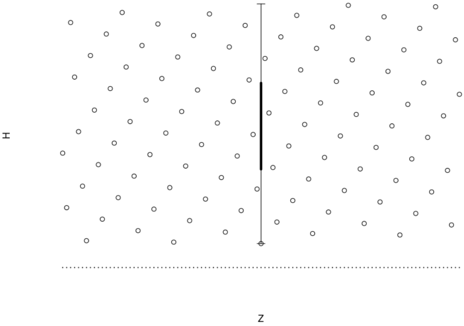

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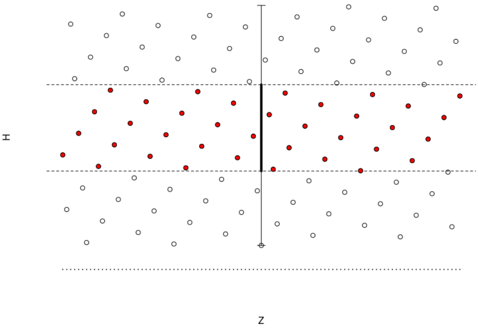
What you see through it:
$\Phi(0) = \mathcal{L} \cap (\mathbb{Z} \times W)$



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The projection of this set to $\mathbb{Z}$:

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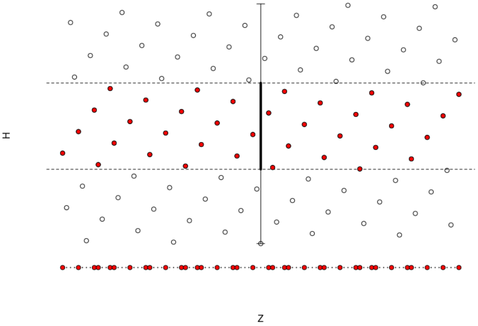

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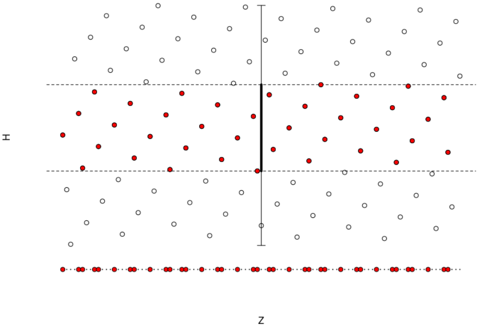

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- $X := \overline{\varphi(H)} \subseteq \{0,1\}^{\mathbb{Z}}$, $\eta := \varphi(0)$, $X_\eta := \overline{\{S^n\eta : n \in \mathbb{Z}\}} \subseteq X$

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Proof of the theorem:

a) $$C_\varphi = \{h \in H : h + \Delta(n) \notin \partial W \ \forall n \in \mathbb{Z}\} = \bigcap_{n\in\mathbb{Z}} ((\partial W)^c - \Delta(n))$$ countable intersection of open dense sets

b) $\emptyset \neq A \subseteq \overline{\mathrm{graph}(\varphi)}$ closed invariant $\Rightarrow \emptyset \neq \pi^H(A) \subseteq X$ closed invariant $\Rightarrow \pi^H(A) = H \supseteq C_\varphi \Rightarrow A \supseteq \mathrm{graph}(\varphi|_{C_\varphi}) \Rightarrow A \supseteq \overline{\mathrm{graph}(\varphi|_{C_\varphi})} =: A_{min}$

c) $\emptyset \neq B \subseteq X$ closed invariant $\Rightarrow \emptyset \neq (\pi^{\{0,1\}^{\mathbb{Z}}})^{-1}(B) \subseteq \overline{\mathrm{graph}(\varphi)}$ closed invariant $\Rightarrow (\pi^{\{0,1\}^{\mathbb{Z}}})^{-1}(B) \supset A_{min} \Rightarrow B \supseteq \pi^{\{0,1\}^{\mathbb{Z}}}(A_{min}) = \overline{\varphi(C_\varphi)}$

d) $\mathring{W} = \emptyset \Leftrightarrow W = \partial W \Leftrightarrow C_\varphi = \bigcap_{k\in\mathbb{Z}} (W^c - \Delta(k)) \Leftrightarrow \varphi(C_\varphi) = \{\underline{0}\}$

Maximal equicontinuous (generic) factors

- $H_W := \{h \in H : W + h = W\}$ and $H_{\mathring{W}} := \{h \in H : \mathring{W} + h = \mathring{W}\} = H_{\overline{\mathring{W}}}$ are closed subgroups of H

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Theorem 2

(KR) $(H/H_{\mathring{W}}, T)$ is the MaxEqcFac of (X, S) and (X_η, S).

(K,KR) $(H/H_W, T)$ is the MaxEqcGenFac of (X_η, S). (Under construction!)

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Theorem 3

(KKL) a) H_W is trivial, hence (H, T) is the MaxEqcGenFac of (X_η, S).
b) $H_{\mathring{W}}$ is trivial if and only if an arithmetic condition (*) on $\mathcal{B}$ holds.
In that case (H, T) is the MaxEqcFac of (X, S) and (X_η, S).

Windows with periodic interior

$\mathcal{B} = \{b_1, b_2, \ldots\}$. Let $S_k := \{b_1, \ldots, b_k\}$, $s_k := \operatorname{lcm}(S_k)$,

$d_k := \lim_{j\to\infty} \gcd(s_k, c_{k+j})$, where $c_k :=$ minimal period of $\bigcup_{b\in\mathcal{B}} \gcd(b, s_k)\cdot\mathbb{Z}$

Proposition (KKL)

a)

$$0 \to H_{\mathring{W}} \to H \cong \varprojlim \mathbb{Z}/s_k\mathbb{Z} \to \varprojlim \mathbb{Z}/d_k\mathbb{Z} \to 0$$

is an exact sequence.

b) $H_{\mathring{W}} \cong \varprojlim \mathbb{Z}/\frac{s_k}{d_k}\mathbb{Z}$.

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Theorem 4 (KR, KKL)

$(\varprojlim \mathbb{Z}/d_k\mathbb{Z}, T_{(1,1,\dots)})$ is the maximal equicontinuous factor of (X_η, S).

$\mathcal{B}$-free systems: Properties of $\mathcal{B}$ and regularity of W

Theorem 5 (KKL)

Assume that $\overline{\Delta(\mathbb{Z}) \cap W} = W$ (which is always true if the set $\mathcal{B}$ is taut). The following are equivalent.

(1) There are no $d \in \mathbb{N}_0$ and no infinite pairwise coprime set $\mathcal{A} \subseteq \mathbb{N}_0 \setminus \{1\}$ such that $d \cdot \mathcal{A} \subseteq \mathcal{B}$.

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(3) $\eta = \varphi(0)$ is a Toeplitz sequence different from $(\dots, 0, 0, 0, \dots)$.

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Theorem 7 (BKKL, KR)

$(X, m_H \circ \varphi_W^{-1}, S)$ is isomorphic to (H, m_H, T).